Comment on "Fractional quantum mechanics" [Phys. Rev. E 62, 3135 (2000)] and "Fractional Schrödinger equation" [Phys. Rev. E 66, 056108 (2002)]

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In this comment, we point out some shortcomings in two papers "Fractional quantum mechanics" [Phys. Rev. E 62, 3135 (2000)] and "Fractional Schrödinger equation" [Phys. Rev. E 66, 056108 (2002)]. We prove that the fractional uncertainty relation does not hold generally. The probability continuity equation in fractional quantum mechanics has a missing source term, which leads to a new phenomenon of probability transportation, probability teleportation, i.e., the probability of a particle is conserved totally but not locally. Since the relativistic kinetic energy can be viewed as an approximate realization of the fractional kinetic energy, the probability teleportation should be an observable relativistic effect in quantum mechanics. With the help of this concept, superconductivity could be viewed as the teleportation of electrons from one side of a superconductor to another and superfluidity could be viewed as the teleportation of helium atoms from one end of a capillary tube to the other.

I. Introduction

In papers [1,2], the standard quantum mechanics [3] is generalized to fractional quantum mechanics. The Schrödinger equation is rewritten as

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) &= H_\alpha\psi(\mathbf{r},t) \\ H_\alpha &= T_\alpha + V = D_\alpha\,|\mathbf{p}|^\alpha + V(\mathbf{r}). \end{aligned} \tag{1}$$

As usual, $\psi(\mathbf{r},t)$ is a wavefunction defined in the 3 dimensional space and dependent on time *t*, D_α is a constant dependent on the fractional parameter $1<\alpha\le 2$, $\mathbf{r},\mathbf{p}$ are the position and momentum operators, respectively, $\hbar$ is the Plank constant, and m is the mass of a particle. The fractional Hamiltonian operator H_α is the sum of the fractional kinetic energy T_α and the potential energy $V(\mathbf{r})$.

When $\alpha = 2$, taking $D_2 = 1/(2m)$, the fractional kinetic energy becomes the classical kinetic energy

$$T_2 = \frac{\mathbf{p}^2}{2m} = T \tag{2}$$

and the fractional Schrödinger equation becomes the standard Schrödinger equation. When $1 < \alpha < 2$, the fractional kinetic energy operator is defined by the momentum representation [1].

However, there exist three shortcomings in this new quantum theory:

(1) The Heisenberg uncertainty relation was generalized to the fractional uncertainty relation [1]

$$<|\Delta x|^{\mu}>^{1/\mu} <|\Delta p|^{\mu}>^{1/\mu} \;>\; \frac{\hbar}{(2\alpha)^{1/\mu}}, \quad \mu < \alpha, 1 < \alpha \le 2 . \tag{3}$$

It seems unsuitable to call this inequality fractional uncertainty relation, and this inequality does not hold mathematically.

(2) Laskin obtained [2] that the fractional probability continuity equation was

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j}_\alpha = 0 \tag{4}$$

where the probability density and current density were defined as

$$\begin{aligned} &\rho = \psi^* \psi \\ &\mathbf{j}_\alpha = -iD_\alpha \hbar^{\alpha-1}\left(\psi^*(-\nabla^2)^{\alpha/2-1}\nabla\psi - \psi(-\nabla^2)^{\alpha/2-1}\nabla\psi^*\right). \end{aligned} \tag{5}$$

In fact, a source term was missing, which indicates a new way of probability transportation, probability teleportation.

(3) The relationship between fractional quantum mechanics and the real world was not given, and it was almost impossible to find the applications of this new theory. Here we will point out that the relativistic kinetic energy can be viewed as an approximate realization of the fractional kinetic energy, which makes the probability teleportation a practical phenomenon.

Now we will discuss these shortcomings in order.

II. On the fractional uncertainty relation

1. The uncertainty relation is independent of wave equations

For simplicity, we do not consider the wavefunction which is not square integrable.

Suppose that $\psi(x)$ is a normalized square integrable wavefunction defined on the x-axis. Heisenberg's uncertainty relation says [3]

$$\sqrt{<(\Delta x)^2>}\;\sqrt{<(\Delta p)^2>}\;\geq\;\frac{\hbar}{2}\quad, \tag{6}$$

where

$$\Delta x = x - <x>,\;\; \Delta p = p - <p>. \tag{7}$$

As usual, *x, p* stand for the 1D position and momentum operator, and $<x>$, $<p>$ stand for their averages on the wavefunction $\psi(x)$, for example,

$$<p>=\int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)dx\,. \tag{8}$$

This relation holds for all the square integral functions, and it is a property of the space of square-integrable functions. A complete mathematical proof can be seen in [4].

As a kinetic equation, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=H\psi(x,t) \tag{9}$$

tells us how to determine the wavefunction-time relation $\psi(x,t)$ by the Hamiltonian operator H given the initial wavefunction $\psi(x,0)$. From the viewpoint of geometry, the equation (9) defines a curve in the space of square integrable functions, which passes a given point at time t=0. Heisenberg's uncertainty relation and the Schrödinger equation are independent.

Laskin generalized the Schrödinger equation, but wave functions remains square-integrable functions, in other words the used function space remains the space of the square-integrable functions, so the Heisenberg uncertainty relation remains true, regardless of the standard or fractional quantum mechanics.

In addition, suppose that there is an uncertainty relation which holds for all the solutions of the fractional Schrödinger equation (1) with certain α , e.g. $\alpha = 1.5$. Since the initial wavefunction $\psi(x,0)$ is an arbitrary square integrable function, we know that this uncertainty relation holds for the whole space of square-integrable functions. Therefore there does not exist a so-called fractional uncertainty relation.

Generally speaking, a generalization of the Schrödinger equation does not generate new uncertainty relations if the wavefunctions remains square-integrable functions.

2. Fractional uncertainty relation does not hold in mathematics

Even with the Levy wave packet (Equation 35 in [1]), the uncertainty relation (3) does not hold in the sense of mathematics. We prove it by contradiction. There are two steps.

(1) Let us consider the case $\mu = 1$ and $\alpha = 1$ first.

The Levy wave packet with $\nu = 1$ at *t*=0 is

$$\begin{aligned}\psi_L(x,0) &= \frac{1}{2\hbar}\sqrt{\frac{l}{\pi}}\int_{-\infty}^{\infty}\exp(-\frac{|p-p_0|l}{2\hbar})\exp(i\frac{p}{\hbar}x)dp \\ &= \frac{1}{2}\sqrt{\frac{l^3}{\pi}}\frac{1}{x^2+(l/2)^2}\exp(i\frac{p_0}{\hbar}x).\end{aligned} \tag{10}$$

The letters *L* means the Levy wave packet and *l* is a reference length.

The related quantities can be calculated as

$$\begin{aligned}&< x >= 0,\ <\text{p}> = p_0, \\ &<|\Delta x|>=<|x|>=\int_{-\infty}^{\infty}|x|\psi_L^*(x,0)\psi_L(x,0)dx = \frac{l}{\pi}, \\ &<|\Delta p|>=<|p-p_0|>=\frac{l}{2\hbar}\int_{-\infty}^{\infty}|p-p_0|\exp(-\frac{|p-p_0|l}{\hbar})dp = \frac{\hbar}{l}.\end{aligned} \tag{11}$$

Therefore we have the inequality

$$<|\Delta x|> \ <|\Delta p|> \ = \ \frac{l}{\pi}\frac{\hbar}{l} \ = \ \frac{\hbar}{\pi} \ < \ \frac{\hbar}{2} \ . \tag{12}$$

(2) Keep $\mu = 1$ as a constant and let $\alpha \to 1^+$.

Since $\nu = \alpha$, the two sides of the inequality (3) are continuous functions about α. Taking $\lim_{\alpha \to 1^+}$ of the both sides of the fractional uncertainty relation (3), we get

$$<|\Delta x|> \ <|\Delta p|> \ \geq \ \frac{\hbar}{2} \tag{13}$$

which contradicts inequality (12). Therefore the fractional uncertainty relation (3) does not hold mathematically.

Further, once the fractional uncertainty relation does not hold for certain α at *t*=0, we know that there exists a small time neighborhood [0,δ), the relation does not hold either, since the wave packet has not expanded very much. In short, the fractional generalization of the Heisenberg uncertainty relation does not hold generally.

We would like to explain why we let $\alpha = 1$, which was not allowed in [1,2]. The case $\alpha = 1$ is just a step of our proof, like an auxiliary line used in geometry problems. Here we add two points. 1) There exist papers that allow $0 < \alpha \le 2$. In [5], Jeng et al ever claimed that Laskin's solutions for the infinite square well problem were wrong by means of the evidence from the case $0 < \alpha < 1$. In fact the evidence from the case $\alpha = 1$ is more straightforward [6]. (2) The fractional Schrödinger equation with $\alpha = 1$ has many closed-form solutions [7], which is an easy starting point for the study of the fractional Schrödinger equation with $1 < \alpha < 2$.

III. On the probability continuity equation

In this section, we will present the correct probability continuity equations in the fractional quantum mechanics, and reveal a new phenomenon of the probability transportation.

1. The correct probability continuity equation

From the fraction Schrödinger equation (1), we can get

$$i\hbar \frac{\partial}{\partial t}(\psi^* \psi) = \psi^* T_\alpha \psi - \psi T_\alpha \psi^* . \tag{14}$$

According to Laskin's definitions of the probability density and the current density(5), the correct probability continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j}_\alpha = I_\alpha \tag{15}$$

has an extra source term

$$I_\alpha = -iD_\alpha \hbar^{\alpha-1}\left(\nabla\psi^*(-\nabla^2)^{\alpha/2-1}\nabla\psi - \nabla\psi(-\nabla^2)^{\alpha/2-1}\nabla\psi^*\right). \tag{16}$$

Specifically, if $I_\alpha(\mathbf{r},t) > 0$, there is a source at position $\mathbf{r}$ and time t, which generates the probability; when $I_\alpha(\mathbf{r},t) < 0$, there is a sink at position $\mathbf{r}$ and time t, which destroys the probability.

It is easy to find the cases where the source term is not zero. For example, take the wave function

$$\begin{aligned} &\psi = \psi_1 + \psi_2 \\ &\psi_1(x,t) = \exp(ik_1 x)\exp(-iE_1 t) \\ &\psi_2(x,t) = \exp(ik_2 x)\exp(-iE_2 t) \end{aligned} \tag{17}$$

with $k_1 > k_2 > 0, E_1 = D_\alpha(\hbar k_1)^\alpha, E_2 = D_\alpha(\hbar k_2)^\alpha$, which is a superposition of two solutions to the fractional Schrödinger equation for a free particle.

We have

$$
\begin{aligned}
I_\alpha &= -iD_\alpha\hbar^{\alpha-1}\left(\nabla\psi^*(-\nabla^2)^{\alpha/2-1}\nabla\psi - \nabla\psi(-\nabla^2)^{\alpha/2-1}\nabla\psi^*\right)\\
&= -iD_\alpha\hbar^{\alpha-1}\left(\left(\psi_1^*+\psi_2^*\right)'(-\nabla^2)^{\alpha/2-1}\left(\psi_1+\psi_2\right)' - \left(\psi_1+\psi_2\right)'(-\nabla^2)^{\alpha/2-1}\left(\psi_1^*+\psi_2^*\right)'\right)\\
&= -iD_\alpha\hbar^{\alpha-1}\left(\psi_1^{*'}(-\nabla^2)^{\alpha/2-1}\psi_2' + \psi_2^{*'}(-\nabla^2)^{\alpha/2-1}\psi_1' - \psi_1'(-\nabla^2)^{\alpha/2-1}\psi_2^{*'} - \psi_2'(-\nabla^2)^{\alpha/2-1}\psi_1^{*'}\right) \qquad (18)\\
&= -iD_\alpha\hbar^{\alpha-1}\left(k_1k_2^{\alpha-1}\psi_1^*\psi_2 + k_1^{\alpha-1}k_2\psi_2^*\psi_1 - k_1k_2^{\alpha-1}\psi_1\psi_2^* - k_1^{\alpha-1}k_2\psi_2\psi_1^*\right)\\
&= -iD_\alpha\hbar^{\alpha-1}(k_1k_2^{\alpha-1} - k_1^{\alpha-1}k_2)(\psi_1^*\psi_2 - \psi_1\psi_2^*)\\
&= 2D_\alpha\hbar^{\alpha-1}(k_1k_2^{\alpha-1} - k_1^{\alpha-1}k_2)\sin\left((k_2-k_1)x - (E_2-E_1)t/\hbar\right),
\end{aligned}
$$

which is not zero unless $\alpha = 2$.

The source term indicates that the probability is no longer locally conserved. As Laskin proved in [8], the total probability in the whole space is conserved. Here is the new picture of the probability transportation in the fractional quantum mechanics: some probabilities can disappear at a region and simultaneously appear at other regions but the total probability does not change. In other words, some probabilities can be teleported from one place to another. Furthermore, if the particle has mass and charge, probability teleportation will imply mass teleportation and charge teleportation. We need to pay close attention to this new phenomenon, mass teleportation contradicts our life experience and charge teleportation contradicts the classical electrodynamics.

2. The case $I_\alpha(\mathbf{r},t) = 0$

When $\alpha = 2$, it is easy to see $I_2(\mathbf{r},t) = 0$. The fractional continuity equation recovers the standard continuity equation.

Proposition. For $1 < \alpha < 2$, we have $I_\alpha(\mathbf{r},t) = 0$ for a free particle with a definite kinetic energy.

Proof.

Since $V(\mathbf{r}) = 0$, the fractional Schrödinger equation is

$$
i\hbar\frac{\partial}{\partial t}\psi\left(\mathbf{r},t\right) = T_\alpha\psi\left(\mathbf{r},t\right). \qquad (19)
$$

For a definite energy E, its solution is

$$
\psi\left(\mathbf{r},t\right) = \int_\Omega C(\theta,\phi)\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\sin\theta d\theta d\phi \exp(-iEt/\hbar), \qquad (20)
$$

$$
E = D_\alpha(\hbar k)^\alpha. \qquad (21)
$$

where Ω is the unit sphere, (k,θ,ϕ) is the spherical coordinate of the vector $\mathbf{k}$, and $C(\theta,\phi)$ is an arbitrary function.

Thus we have

$$(-\nabla^2)^{\alpha/2-1}\psi(\mathbf{r},t)=k^{\alpha-2}\psi(\mathbf{r},t) \tag{22}$$

$$(-\nabla^2)^{\alpha/2-1}\psi^*(\mathbf{r},t)=k^{\alpha-2}\psi^*(\mathbf{r},t)\,. \tag{23}$$

In this case the source term vanishes

$$\begin{aligned}
I_\alpha &= -iD_\alpha\hbar^{\alpha-1}\left(\nabla\psi^*(-\nabla^2)^{\alpha/2-1}\nabla\psi-\nabla\psi(-\nabla^2)^{\alpha/2-1}\nabla\psi^*\right)\\
&= -iD_\alpha\hbar^{\alpha-1}\left(\nabla\psi^*\nabla(-\nabla^2)^{\alpha/2-1}\psi-\nabla\psi\nabla(-\nabla^2)^{\alpha/2-1}\psi^*\right)\\
&= -iD_\alpha\hbar^{\alpha-1}k^{\alpha-2}\left(\nabla\psi^*\nabla\psi-\nabla\psi\nabla\psi^*\right)=0,
\end{aligned} \tag{24}$$

and the continuity equation has a sourceless form

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\mathbf{j}_\alpha=0\,, \tag{25}$$

with

$$\begin{aligned}
&\rho=\psi^*\psi\\
&\mathbf{j}_\alpha=-iD_\alpha\hbar^{\alpha-1}k^{\alpha-2}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right).
\end{aligned} \tag{26}$$

This completes the proof.

We emphasize that in scatter problems the source term $I_{1<\alpha<2}(\mathbf{r},t)\neq 0$ at the detector's location, though the potential at the detector may be zero. There are two reasons:1) The particle is not free so the relation (22) does not hold, and 2) the kinetic energy of particles from the scattering source may not be exactly the same, i.e. they may not be strictly monoenergetic.

How to develop a scattering model based on the correct continuity equation (15) is a valuable problem in fractional quantum mechanics.

IV. On the realization of the fractional quantum mechanics

In papers [1, 2], the relationship between the fractional quantum mechanics and the real world was not given. A natural question is, which particle has a fractional kinetic energy. If there are no fractional particles in our world, why do we need fractional quantum mechanics? To relate the fractional quantum mechanics to the real world, we regard the relativistic quantum mechanics [9-11] as an approximate realization of fractional quantum mechanics.

According to the special relativity, the relativistic kinetic energy is

$$T_r = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \ , \tag{27}$$

where the subscript r means special relativity.

For the case of low speed, the relativistic kinetic energy is approximately the summation of the rest energy and the classical kinetic energy ($\alpha = 2$)

$$T_r \approx mc^2 + \frac{\mathbf{p}^2}{2m} = mc^2 + T_2 \ , \tag{28}$$

and for the case of extremely high speed, where the rest energy can be neglected, the relativistic kinetic energy is the fractional kinetic energy with $\alpha = 1$

$$T_r \approx |\mathbf{p}| c = T_1 \ . \tag{29}$$

Generally speaking, if the speed of a particle increases from low to high, the relativistic kinetic energy T_r will approximately correspond to a fractional kinetic energy T_α, whose parameter α changes from 2 to 1. Therefore the relativistic kinetic energy is an approximate realization of the fractional kinetic energy.

The Hamiltonian function with the relativistic kinetic energy is [9-10]

$$H_r = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} + V(\mathbf{r}) . \tag{30}$$

Historically, using this Hamiltonian function, Sommerfeld calculated the relativistic correction to Bohr's hydrogen energy levels, so that the fine structure in the hydrogen spectrum was explained exactly [3].

The relativistic Schrödinger equation is [9-10]

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H_r \psi(\mathbf{r}, t) . \tag{31}$$

Using the perturbation method, we recently calculated the relativistic correction to the hydrogen energy levels from the Schrödinger equation. The resultant energy levels contain an α^5 term, which can explain the Lamb shift at an accuracy of 41% [12,13].

The continuity equation is now

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j}_r = I_r , \tag{32}$$

with the current density and the source term

$$\mathbf{j}_r = -\frac{1}{i\hbar}(\psi^* T_r \nabla^{-2}\nabla\psi - \psi T_r \nabla^{-2}\nabla\psi^*),$$
$$I_r = -\frac{1}{i\hbar}(\nabla\psi^* T_\alpha \nabla^{-2}\nabla\psi - \nabla\psi T_\alpha \nabla^{-2}\nabla\psi^*). \tag{33}$$

Again, the probability is not locally conserved, but the total probability in the whole space is conserved

$$i\hbar\frac{\partial}{\partial t}\int_{R^3}\psi^*\psi d^3\mathbf{r} = \int_{R^3}\psi^* T_r\psi d^3\mathbf{r} - \int_{R^3}\psi T_r\psi^* d^3\mathbf{r} = 0\,. \tag{34}$$

Similarly, for a free particle with a definite kinetic energy, we have $I_r = 0$.

Since the relativistic kinetic energy is true and the classical kinetic energy is approximate, the probability continuity equation with the source term (32) is true and the popular probability continuity equation in standard quantum mechanics [3] is approximate.

Therefore we need to base our scattering model on the continuity equation with the source term, i.e. Eq. (32), calculate the variation between the new model and the traditional model, and design experiments to observe the phenomenon of the probability teleportation.

Since the relativistic Schrödinger equation (31) is not relativistically covariant [9,10], violates the causality [14], and is non-local [15] and complicated [9], the researchers on this equation have been criticized since the early days of quantum mechanics. A positive experimental result on the probability teleportation will end this situation ultimately.

VI. Conclusion

We prove that the fractional uncertainty relation does not hold generally. The missing source term in the fractional probability continuity equation leads to a new phenomenon of the probability transportation, probability teleportation. According to the special relativity, the classical kinetic energy is just an approximation in the low speed case, so it would be a hopeful direction to study the phenomenon of the probability teleportation in scattering theory and experiments. What is more, the concept in this comment brings us a very intuitive explanation for superconductivity and superfluidity.

Furthermore, we would like to mention some closely related topics. For fractional quantum field theory, see [16], and for quantum mechanics based on a general kinetic energy, see [17].

By the way, the symbol H^+ in [1, 2] seems to be H.

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Academic cooperation on the probability teleportation is welcome, including theoretical studies and experimental designs, as well as joint funding applications, seminars, conferences, and special issues.

Appendix. An intuitive explanation for the superfluidity and superconductivity based on teleportation.

In [18] the authors applied the fractional quantum mechanics to explain some property of the superfluid ^{4}He.

From fractional quantum mechanics, they correctly got the fractional probability continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j}_\alpha = I_\alpha , \tag{35}$$

Where ρ is viewed as the density of the superfluid, $\mathbf{j}_\alpha$ as the mass current density, and I_α as extra sources.

In order to keep consistent with the well-known fluid continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0 , \tag{36}$$

the author claimed that in the superfluid $I_\alpha \approx 0$, since the wavefunction describing the He atoms can be assumed to be

$$\psi(\mathbf{r}) = \sum_{\mathbf{p}} C_{\mathbf{p}} \exp\left(i\mathbf{p}\cdot\mathbf{r}/\hbar\right) , \tag{37}$$

where the summation goes over the momentums with approximately equal $|\mathbf{p}|$.

This is very difficult to understand. (1) We do not know why the atoms are on such a special state. (2) Since the potential $V(\mathbf{r})$ is not zero or a constant, (see Eq. (13) in [18]), the wavefunction (37) is not an eigenfunction, so even if at t=0 the momentum magnitudes $|\mathbf{p}|$ are the same, they will become different soon.

On the contrary, we think that the source term I_α or I_r is not zero generally. Here is our superfluid model based on equation (35) or (32). When the superfluid ^{4}He is still, every atom can disappear at one place and appear in another distant place. This explains why heat can be conducted easily by the superfluid. When the superfluid flows, some superfluid moves forward normally, which has friction and some is teleported from one place to another place, which has no friction. Thus we do not need to artificially divide the superfluid into a normal component and a superfluid component.

Now it becomes an urgent task to observe whether the mass teleportation really exists in the superfluid experiment. An easy way is to measure the velocity of the superfluid in the capillary tube and the mass coming out from the tube to check whether they are consistent. We are working in this direction.

Similarly, based on the concept of teleportation, the superconductivity can be viewed as such an effect that some electrons are teleported from one side of the superconductor to the other side; of course they do not dissipate energy. Currently, physicists say that the superconducting electrons pass through the Josephson junction by quantum tunneling, but an open question is why the supercurrent can flow through the non-superconducting metal in a SNS junction **without dissipation**. (How can one drive through a toll tunnel without payment?) The straightforward explanation is that electrons are teleported from one side of the junction to the other. Let us work together to complete this most intuitive mechanism in the theory and on experiments.

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