略谈国内高中数学所谓分析法和综合法

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1 内容提要

(1) 国内高中教材中所谓的的“分析法”和“综合法”的不等式证明方法，分别源自二千多年前的古希腊的“analysis”和“synthesis”，但与之意义并不完全相同。

(2) 术语“analytic proof”在现代数学中已有完全不同的含义。“synthetic proof”已是不常用的术语，即使被人应用也含义不一，且与国内教材中的用法很不相同。

(3) “analytic method”和“synthetic method”都是多义的词语，其古希腊含义在现代并不常用，建议国内高中教材或参考书在使用它们时应当慎重。

2 引子

在数学家蔡天新教授的《数学与艺术》第一章第 2 节“《诗学》与《原本》”中有这么一段话 (颜色均为笔者所加)：

> 虽然柏拉图本人并没有在数学研究方面作出特别突出的贡献 (有人将分析法和归谬法归功于他)，但他的学园却是那个时代希腊数学活动的中心，大多数重要的数学成就均由他的弟子取得。

蔡教授对分析法一词做了一个注释：

> 分析法是把复杂的事物或现象分解成若干简单的组成部分，分别进行研究的方法。分析法是综合法的对称，后者是把各个组成部分、各个方面和各种因素联系起来，从总体上认识和把握事物或现象的方法。

分析法和综合法正是本文的主题 (前面标红的归谬法也会再提到) 。那么，柏拉图本人或者他的弟子们发明/发现的分析法是不是蔡教授所说的那样呢？它跟高中数学又有什么关系呢？这要从笔者的困惑谈起。

3 笔者的困惑

笔者从高中时代开始就有个百思不解的问题：在高中数学的**不等式证明**中，有所谓“综合法”和“分析法”，为什么在做其他题目时就基本不提了？

笔者在以后多年的高等数学、物理和化学等理科学习过程中，以及在工作过程中，也都很少再见到它们的影子(国内高中数学书除外)。

笔者亦曾就此询问远离高中数学的朋友，发现他们对高考不等式中所谓的“分析法”和“综合法”早已不知所云(阅读本文的读者大概也多是如此)，对反证法、数学归纳法之类则仍有印象。这并不让人意外，因为国内高中数学书里的所谓“分析法”和“综合法”，与其在日常语言和现代数学语言中的含义都差异甚大(前面蔡天新教授的文字就是证明)。

现在来分别看一个所谓“分析法”和“综合法”证明不等式的例子，分别见图1和图2。它们都是截图自高中数学 A 版选修 4-5《不等式选讲》(人民教育出版社，2019 年印刷)。教材中说，“分析法”是执果索因的思考和证明方法，“综合法”又叫顺推证法或由因导果法。

证明命题时，我们还常常从要证的结论出发，逐步寻求使它成立的充分条件，直至所需条件为已知条件或一个明显成立的事实（定义、公理或已证明的定理、性质等），从而得出要证的命题成立，这种证明方法叫做**分析法**（analytical method）．这是一种执果索因的思考和证明方法．

例 3 求证$\sqrt{2}+\sqrt{7}<\sqrt{3}+\sqrt{6}$.

分析：从不等式的结构不易发现需要用哪些不等式的性质或事实解决这个问题，因此用分析法．

证明：因为$\sqrt{2}+\sqrt{7}$和$\sqrt{3}+\sqrt{6}$都是正数，所以要证

$$\sqrt{2}+\sqrt{7}<\sqrt{3}+\sqrt{6},$$

只需证

$$(\sqrt{2}+\sqrt{7})^2<(\sqrt{3}+\sqrt{6})^2.$$

展开得

$$9+2\sqrt{14}<9+2\sqrt{18},$$

只需证

$$\sqrt{14}<\sqrt{18},$$

只需证

$$14<18.$$

因为 $14<18$ 成立，所以$\sqrt{2}+\sqrt{7}<\sqrt{3}+\sqrt{6}$成立．

图 1: 分析法证明实例

例 1 已知 a，b，$c>0$，且不全相等，求证

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)>6abc.$$

分析：观察欲证不等式的特点，左边 3 项每一项都是两个数的平方之和与另一个数之积，右边是三个数的积的 6 倍．这种结构特点启发我们采用如下方法．

证明：因为 $b^2+c^2\geqslant 2bc$，$a>0$，所以

$$a(b^2+c^2)\geqslant 2abc. \quad ①$$

因为 $c^2+a^2\geqslant 2ac$，$b>0$，所以

$$b(c^2+a^2)\geqslant 2abc. \quad ②$$

因为 $a^2+b^2\geqslant 2ab$，$c>0$，所以

$$c(a^2+b^2)\geqslant 2abc. \quad ③$$

由于 a，b，c 不全相等，所以上述①②③式中至少有一个不取等号，把它们相加得

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)>6abc.$$

一般地，从已知条件出发，利用定义、公理、定理、性质等，经过一系列的推理、论证而得出命题成立，这种证明方法叫做综合法（synthetical method）．综合法又叫顺推证法或由因导果法．

图 2: 综合法证明实例

在这里，“分析法”和“综合法”这两个词的意思显然不同于其现代汉语中的日常意义，那么，上述意义的来源是什么呢？其初始含义又是什么呢？

事实上，这两个词是两个非常古老的术语。它们当然也像大多数数学术语一样，都不是中国古代数学词汇，而是舶来品。

没错，我们要回到蔡天新教授在前述引文中所讨论的、遥远的古希腊时代。

4 西方数学史家的记录

科学史家 Florian Cajori (1859-1930) 在其著作《A History of Mathematics》(数学史) 中介绍希腊数学的部分写道 (Project Gutenberg EBook 版本，第 35 页。所有中文和颜色均为笔者所加，读者可以只关注中文和加颜色的文字)：

> The terms synthesis and analysis（术语“综合”与“分析”）are used in mathematics in a more special sense than in logic. In ancient mathematics they had a different meaning from what they now have（它们在古代数学中的含义与现代不同）.
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> The oldest definition of mathematical analysis as opposed to synthesis (数学上最古老的“分析”和“综合”的定义) is that given in Euclid, XIII. 5（见于欧几里得《几何原本》第 13 卷命题 5。在欧氏原文中应该没有这句话，应是

他人的“注释”), which in all probability was framed by Eudoxus (欧多克索斯，古希腊数学家，欧几里得《几何原本》的不少资料来自他): ‘Analysis is the obtaining of the thing sought by assuming it (先假设它成立) and so reasoning up to an admitted truth (不断推理，直至得到一个公认的真理); synthesis is the obtaining of the thing sought by reasoning up to the inference and proof of it.’ (上述两个定义与国内高中数学参考书中的含义相当。)

The analytic method (分析法) is not conclusive, unless all operations involved in it are known to be reversible (分析法只有每一步都“可逆”时，证明才可靠). To remove all doubt, the Greeks, as a rule, added to the analytic process a synthetic one (综合法), consisting of a reversion of all operations occurring in the analysis (综合法与分析法的步骤相反). Thus the aim of analysis was to aid in the discovery of synthetic proofs or solutions (“分析”的目标，是帮助发现“综合法”的证明或方法。在国内高中数学书中也常说，用“分析法”找不等式的证明思路，用“综合法”书写过程。).

另一个数学史家 James Gow (1854–1923) 在其名著《A Short History of Greek Mathematics (希腊数学简史)》(Cambridge Library Collection 版本，第 177 页) 则把“综合”和“分析”的过程说得更清楚:

In other words, the synthetic proof (“综合法”证明) proceeds by shewing that certain admitted truths involve the proposed new truth (揭示已知真理中蕴含待证的新真理): the analytic proof (“分析法”证明) proceeds by shewing that the proposed new truth involves certain admitted truths (揭示待证新的新真理中蕴含了已知真理). An analytic proof begins by an assumption, upon which a synthetic reasoning is founded.

The Greeks distinguished theoretic from problematic analysis (古希腊人区分“定理型的分析”和“问题型的分析”). A theoretic analysis is of the following kind. To prove that A is B (为了证明命题 A 等价于命题 B), assume first that A is B (先假设 A 等价于 B). If so, then, since B is C and C is D and D is E, therefore A is E. If this be known a falsity, A is not B. But if this be a known truth and all the intermediate propositions be convertible (以上是“分析”过程，以下是相反的“综合证明”过程), then the reverse process, A is E, E is D, D is C, C is B, therefore A is B, constitutes a synthetic proof of the original theorem.

读者可在维基百科“Analysis”词条中(网址：https://en.wikipedia.org/wiki/Analysis) 看到上面的两段话（不过，维基百科的引用不完全准确）。它们解释了 analysis(分析) 与 synthesis(综合) 的古老含义与来历。国内高中数学书中所谓的“分析法”和“综合法”，使用的就是上面两段引文中的含义。

然而，这并不是全部。我们再来看看著名的数学史家 Heath 爵士在其经典英译《The Thirteen Books of the Elements》Vol. 1 (《原本》第一卷》) 里的说法。

Heath 花了近 6 页的篇幅专门介绍“Analysis and Synthesis”这两个术语 (在其导言部分第 IX 章)，里面除了有类似上述引文的内容之外，还有这么一段引文 (原文无分段，据称它源自公元 3-4 世纪的 Pappus)：

> Now analysis is of two kinds, the one directed to searching for the truth and called theoretical (定理型的), the other directed to finding what we are told to find and called problematical (问题型的).
>
> (1) In the theoretical kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hyhypothesis, to something admitted: then (a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis (情形 (a) 与前面引文相同), but (b), if we come upon something admittedly false, that which is sought will also be false. (情形 (b) 却不同：如果得到错误的结论，表明所假设之事是错误的)
>
> (2) (以下引文解释了什么是“问题型分析”，与本文关系不大) In the problematical kind we assume that which is propounded as if it were known, after which we pass through its successive consequences, taking them as true, up to something admitted: if then (a) what is admitted is possible and obtainable, that is, what mathematicians call given (“给定”，比如给定三条线段的长度), what was originally proposed will also be possible, and the proof will again correspond in reverse order to the analysis, but if (b) we come upon something admittedly impossible, the problem will also be impossible. (在几

何命题中，问题型的分析是为了解决一个作图问题，定理型的分析则是为了证明一个定理。[①])

由上面if we come upon something admittedly false 这半句话可知，在古希腊人那里，现在所谓的“反证法”也是“分析法”的一种。

事实上，Heath 爵士还有以下一段说明 (ibid)：

Reductio ad absurdum a variety of analysis. (归谬法也是一种分析法)

In the process of analysis starting from the hypothesis that a proposition A is true and passing through B, C... as successive consequences we may arrive at a proposition K which, instead of being admittedly true, is either admittedly false or the contradictory of the original hypothesis A or of some one or more of the propositions B, C... intermediate between A and K. (分析过程得到一个错误的命题或与假设矛盾)

Now correct inference from a true proposition cannot lead to a false proposition; and in this case therefore we may at once conclude, without any inquiry whether the various steps in the argument are convertible or not, that the hypothesis A is false, for, if it were true, all the consequences correctly inferred from it would be true and no incompatibility could arise.

This method of proving that a given hypothesis is false furnishes an indirect method of proving (由此产生一种间接证明方法) that a given hypothesis

[①]顺便说一下，最近上海三联书店出版了冯翰翘先生的《The Thirteen Books of the Elements》汉译“全本”，即包括引论和大部分注释在内的版本。对于冯老先生花费如此巨大的精力翻译此 1000 余页巨著，笔者表示由衷的敬佩。有点遗憾的是，在笔者已经阅读的部分中，有不少地方翻译得不太到位，此处引用的这一段就是个例子。比如，冯老先生把 theoretical kind 译为“理论型的”。事实上，problem (命题) 和 theorem (定理) 是古希腊对几何命题的一种基本分类方式。在《The Thirteen Books of the Elements》中，此处引文前几页就有一小节的名称叫做 problems and theorems。Heath 爵士在这一小节则开始时就引用了 Proclus(公元 5 世纪) 的解释：

Again the deductions from the first principles are divided into problems and theorems, the former embracing the generation, division, subtraction or addition of figures, and generally the changes which are brought about in them, the latter exhibiting the essential attributes of each。

冯老先生对此的翻译是：从基本原理开始的演绎分为问题和定理，前者包含作图和图形的分割，减少或增加，一般地，某些变化，后者展示每一个的基本属性。

笔者根据理解对此句翻译如下：从基本原理开始的演绎被区分为问题和定理，前者包括图形的生成、分割、裁减或合并，以及一般地说在它们之中产生的 [各种] 变化，后者则展示 [它们] 每一个的基本属性。

A is true, since we have only to take the contradictory of A and to prove that it is false. This is the method of reductio ad absurdum (归谬法) , which is therefore a variety of analysis.

The contradictory of A, or not-A will generally include more than one case and, in order to prove its falsity, each of the cases must be separately disposed of (如果反面有多种情况, 就要一一驳倒) : e.g., if it is desired to prove that a certain part of a figure is equal to some other part, we take separately the hypotheses (1) that it is greater, (2) that it is less, and prove that each of these hypotheses leads to a conclusion either admittedly false or contradictory to the hypothesis itself or to some one of its consequences. (顺便说一下，按照这里的说法，蔡天新教授对古希腊 "归谬法 (reductio ad absurdum)" 的理解似乎也不正确，见脚注[①])

笔者之所以大段大段地引用有点重复的英文，只是想强调：**在欧洲古代几何学传统中，术语分析和综合有其特殊含义，而且它们不是无足轻重的概念**。事实上，Heath 爵士说 (ibid。请注意，这段话最早出版于 1908 年[②])：

The ancient Analysis has been made the subject of careful studies by several writers during the last half-century, the most complete being those of Hankel, Duhamel and Zeuthen; others by Ofterdinger and Cantor (Moritz Benedikt Cantor) should also be mentioned.

5 现代西方人眼中的"分析法"和"综合法"

再来看看西方人所理解的"分析证明" (analytic proof) 和"综合证明" (synthetic proof) 的现代含义。维基百科"analytic proof"词条 (网址：https://en.wikipedia.

[①] 蔡天新教授对归功于柏拉图的归谬法有如下脚注：

归谬法是反证法的一种形式。用反证法证时，如果命题只出现一种情况，那只需将它驳倒就可以，这种反证法叫"归谬法"。如果有多种情况，那必须将它们一一驳倒，才能证明命题成立，这种反证法叫"穷举法"。

根据 Heath 爵士的说法，蔡天新教授所说的两种情况都叫reductio ad absurdum，后者正对应汉译"归谬法"。"反证法"的现代说法是 proof by contradiction。笔者所识有限，不知道 reductio ad absurdum 与 proof by contradiction 二者之间的关系。

[②] 世界图书出版公司有影印本《欧几里得原理十三本书》，但令人比较遗憾的是，它是采用 1908 年的第一版而非 1925 年的第二版，所以不建议读者购买它。

org/wiki/Analytic_proof) 是这么说的 (笔者对现代数学了解不多，只能借助网络上的文字来理解。另外，笔者查看这些词条的时间已经比较久远了，或许现在它们已被编辑过)：

> In mathematics, an analytic proof is a proof of a theorem in analysis that only makes use of methods from analysis, and which does not predominantly make use of algebraic or geometrical methods. The term was first used by Bernard Bolzano (波尔查诺，1781-1848，捷克数学家), who first provided a non-analytic proof of his intermediate value theorem and then, several years later provided a proof of the theorem which was free from intuitions concerning lines crossing each other at a point, and so he felt happy calling it analytic.

即使我们完全不管 "analytic proof" 的现代含义，单单 "The term was first used by Bernard Bolzano (这个术语最早是由波尔查诺使用的)" 这句话就很能说明问题，现代的 "analytic proof" 是个 19 世纪才开始使用的新术语！它的译名应该是 "解析证明"。这个 "analytic" 与 "解析几何" (analytic geometry) 中的 "analytic" 是相近的。对应地也就有现代意义上的综合几何 ("Synthetic Geometry")，也就是以欧几里得《几何原本》为代表的不用坐标表述的公理化几何或纯几何 (axiomatic 或 pure geometry)。

我们可以看一下名著《高观点下的初等数学》中的描述 (此处引用英译文字 "Felix Klein. *Elementary Mathematics from a Higher Standpoint: Volume II, Geometry*, Springer (2016)"。斜体原文即有)：

> However, I should like to add to this account an explanation of the *difference between analytic and synthetic geometry* (论述解析几何和综合几何之间的区别), which always plays a part in such discussions. According to their original meaning, emphcsynthesis and analysis are different methods of presentation (它们的最初含义是两种不同的表现方式). Synthesis begins with details, and builds up from them more general, and finally the most general notions. Analysis, on the contrary, starts with the most general, and separates out more and more the details (不论克莱因教授的说法是不是它们的最初含义，这里确是蔡天新教授所用的含义). It is precisely this difference in meaning, which finds its expression in the designations synthetic and analytic chem-

istry（“合成化学”与“分析化学”，正由上述含义的差别而来）Likewise, in school geometry, we speak of the *analysis of geometric constructions*: we assume there that the desired triangle has been found, and we then dissect the given problem into separate partial problems.[①]

In higher mathematics, however, these words have, curiously, taken on an entirely different meaning（很奇怪地，它们在高等数学中具有跟上面完全不同的含义）. Synthetic geometry is that which studies figures as such, without recourse to formulas, whereas analytic geometry consistently makes use of such formulas as can be written down after the adoption of an appropriate system of coordinates (综合几何通过图形来研究图形，不借助公式；解析几何则需要采用合适的坐标系后，并由此写出和不断地利用表达式）……

既然有“Synthetic Geometry”，就会有与之相对应的“Synthetic proof”，即不建立坐标系的几何证明方法。随便举一例，搜索“Another Synthetic Proof of Dao's Generalization of the Simson Line Theorem”，可得一篇 2016 年的论文 (http://forumgeom.fau.edu/FG2016volume16/FG201608.pdf）。不必细看该论文，只需观察其中的符号，读者即可明白所谓“Synthetic Proof”就是“纯几何”证明。

但是，“synthetic proof”在维基百科似乎没有关于它的独立词条。

在维基百科的“mathematical proof”(数学证明) 词条中 (网址:https://en.wikipedia.org/wiki/Mathematical_proof) 中也没有“analytic proof”和“synthetic proof”，但是提到了这样一句话：“A classic question in philosophy asks whether mathematical proofs are analytic or synthetic. Kant, who introduced the analytic-synthetic distinction, believed mathematical proofs are synthetic (康德相信“数学证明是综合的”）.”可见，康德哲学里的分析与综合，又与前面据说的分析与综合大不相同。

那么，现代欧美人会不会在类似古希腊的含义下使用“analytic proof”和“synthetic proof”呢？在数十本国外高等数学和中学数学教材或参考书中，笔者很少见

[①]《高观点下的初等数学》第二卷中译本第 66 页对本段的翻译如下：

然而，我还是要一下解析几何与综合几何之间的区别，因为在讨论上总要涉及这种区别。按它们的原来意义，综合和解析是描述的不同方法。综合从细节着手，从而建立一般概念，最后达到更一般的概念。解析则相反，从最一般开始，分离出起来越多的细节。正是这种意义上的差别，造成了合成化学与分析化学的名称的不同。类似地，在中学几何里，我们谈几何结构的分析，总是化异为同已找到所需的三解形，然后把给出的问题分成若干个问题。

到这种情况 (但还是有的，例如在 Springer 出版社的图书《Proof and Proving in Mathematics Education: The 19th ICMI Study》中)。在网上也能检索到类似但又不完全一样的用法。例如，笔者检索到一篇介绍关于证明方法的文章 (网址是https://mathcs.clarku.edu/~djoyce/ma225/writingProofs.pdf)，它是美国克拉克大学 (Clark University) 数学与计算机科学系近世代数 (Modern Algebra) 课的 2017 年秋季课程资料，是关于证明方法介绍的。现将这篇短文中的部分内容抄录如下：

> **Synthetic proofs**. These are the standard proofs that you see in textbooks. They build up to the conclusion one step at a time. You can easily follow a synthetic proof, but it's hard to construct them, except the easiest ones. (我们很容易看懂综合法证明，但不易构造它，特简单的除外)
>
> One kind of synthetic proof is a *direct proof* (“综合法”的一种类型是直接证明). For a direct proof of an implication $P \implies Q$, assume the hypothesis P and derive the conclusion Q. There may intermediate steps. From P you derive R, from R you derive S, and from S you derive the final conclusion Q.
>
> There are other kinds of synthetic proofs such as *indirect proofs and proof by contrapositive* (“综合法”的另一种类型是间接证明和利用“逆否命题”证明). For an *indirect proof*, also called a proof by contradiction (反证法), to prove P, instead assume P is false and derive any contradiction, that is, any statement of the form Q and not Q (这里又把“反证法”看作是一种“综合证明”). For a *proof by contrapositive, to prove $P \implies Q$, you can instead prove the contrapositive*, which is logically equivalent, that is, assume that Q is false and derive that P is false……
>
> **Analytic proofs**. Although it's often easier to find an indirect proof than a direct proof, there's a different kind of proof that's closer to what we do when we're looking for a proof. That is, we start at the end and work back. To prove an implication $P \implies Q$, start with the goal Q and break it down into simpler statements that imply it. You might find that Q follows from S, then S follows from R, and then R follows from P. You've succeeded in showing that $P \implies Q$. You can always turn an analytic proof into a synthetic proof by reversing the order of your statements. It makes it easier

to follow the proof, but it hides the process you used to find the proof. (这个对“分析”的表述，与前面引述的古希腊意义相近，但显然不包括“归谬法”)

上文对“analytic proof”的定义与国内教材相同。但是，它对“synthetic proof”的定义与国内教材、古希腊用法都不相同，因为它把“反证法 (proof by contradiction)”也归入“synthetic proof”。(这种做法有它的道理，既然已经限制了“analytic proof”是“由果及因”，与之相对的“synthetic proof”就可以包括“analytic proof”以外的各种方法)

另外，在现代数学中的 mathematical analysis 即“数学分析”中，analysis 的含义与其古希腊含义显然差异极大，而 synthesis 在现代数学中通常没有特殊的含义。

6 建议慎重使用“综合法”和“分析法”

回到“引子”，读者现在已经知道，蔡天新教授犯了个小小的错误，他用现代思维解释了两个古希腊术语。但换个角度说，或许这并不是蔡教授一个人的问题。试想，在高考过后，绝大部分读者或许还能知道“反证法”“数学归纳法”之类，能记得古希腊意义上的“综合法”“分析法”的人估计就不多了 (以高中数学为业的老师除外)。

所以，笔者在想，在中学阶段，那些用途很窄而且容易产生歧义的术语，是否应当在教材中予以保留？笔者的建议是，应当慎重使用本文所论述的“综合法”和“分析法”。如果一定要使用它们，笔者的建议是：

(1) 尽量厘清“综合法”和“分析法”在不同历史阶段、不同语境下的含义，并让学生们对此有所了解。

(2) 要处理好与其他方法的逻辑层次关系。例如，笔者以为，将“作差法”“分析法与综合法”“反证法与放缩法”等并列 (前面提到的高中数学选修 4-5《不等式选讲》就是这么干的)，就像把动物与袋鼠之类并列一样，是有逻辑问题的。其实不难看出，“作差法”本质上就是一种“分析法”。而前面已经说过，“反证法”在古希腊人那里也是一种“分析法”。至于“放缩法”，我实在想不出，为什么该教材作者会把它与“反证法”并列。

(3) 最好不要把“综合法”和“分析法”二词局限在不等式证明这一狭窄的领域，毕竟它们源自古希腊几何学，因此起码在平面几何证明中介绍它们 (哪怕是作为历史背景) 会是更自然、更符合历史。