Physics Letters A 372 (2008) 5052–5053

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Critical exponents in D dimensions for the Ising model, subsuming Zhang's proposals for $D = 3$

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ARTICLE INFO

Article history:
Received 24 April 2008
Accepted 30 April 2008
Available online 14 June 2008
Communicated by V.M. Agranovich

ABSTRACT

Zhang has recently proposed critical exponents of the Ising model for dimensionality $D = 3$. We have set up a D-dimensional result for the critical exponent $\delta(D)$ which embraces Zhang's value for $D = 3$ as well as known values for $D = 1$, 2 and 4. Scaling relations yield further critical exponents as a function of D. Finally, a critical exponent defined for random walks is treated.

In a very recent study, Zhang [1] has proposed closed results for critical exponents of the three-dimensional Ising model, based on two, as yet unproved, conjectures. Here, we show that Zhang's critical exponent δ for dimensionality $D = 3$ can be utilized, along with known results for $D = 1$, 2 and 4, to allow a form of the critical isotherm exponent $\delta(D)$ to be set up. Combined with known scaling relations [2,3], two other critical exponents then become available as a function of $\delta(D)$. It will be of considerable interest for the future if a fractal model can be solved; ideally exactly but more realistically numerically, to test, and if necessary to refine, the form of $\delta(D)$ constructed in this Letter.

With this brief background, we first stress, by appealing to Table 1 in Zhang's paper [1], that the results collected there for $\delta(1)$ and $\delta(4)$ are embraced by the simple formula $9/(D-1)$. This expression will therefore be refined to fit Onsager's exact result [4] $\delta(2) = 15$ and Zhang's proposal that $\delta(3) = 13/3$. Taking a two-parameter form of $\delta(D)$ that is constructed to yield $\delta(1)$ and $\delta(4)$ also exactly, we shall write

$$\delta(D) = \frac{9}{(D-1)} + \left(f + \frac{g}{D}\right)\left(\frac{1}{D} - 1\right)\left(\frac{1}{D} - \frac{1}{4}\right). \tag{1}$$

Using the values quoted above for $D = 2$ and 3, it readily follows from Eq. (1) that

$$f + \frac{g}{2} = -48 \tag{2}$$

and

$$f + \frac{g}{3} = 3. \tag{3}$$

From Eqs. (2) and (3), the proposed formula (1) for $\delta(D)$ is made precise by inserting $f = 105$ and $g = -306$. Turning next to the scaling hypothesis, Stanley [2] records in Table 11.1, page 185 the scaling relations

$$\alpha + 2\beta + \gamma = 2, \tag{4}$$

together with

$$\alpha + \beta(\delta + 1) = 2. \tag{5}$$

Referring again to Zhang's Table 1, we have $\alpha = 0$ for $D = 2$, 3, and 4, and hence from Eq. (5)

$$\beta(D) = \frac{2}{\delta(D) + 1}. \tag{6}$$

Inserting $\alpha = 0$, again for $D = 2$, 3, and 4, into Eq. (4) readily yields the exponent $\gamma(D)$ as

$$\gamma(D) = 2\bigl(1 - \beta(D)\bigr). \tag{7}$$

Eq. (7) determines $\gamma(D)$ in terms solely of $\delta(D)$ in Eq. (1) when use is made of $\beta(D)$ in Eq. (6). The values 7/4, 5/4 and 1 for γ are then regained for $D = 2$, 3 and 4, respectively.

At this point, we felt it of some interest to draw attention to a less conventional 'critical' exponent related to the theory of random walks. This exponent was considered in the early pioneering studies of Flory [5,6]; see also the refinements of de Gennes [7]. If one considers the mean square extension R^2 for a random, self-excluding walk of N jumps on a D-dimensional lattice, the above critical exponent, $\rho(D)$ say, will be defined here by

$$R^2(D) = \text{constant} \times \left(N^{2\rho(D)}\right). \tag{8}$$

Then the Flory analytic result, denoted below by $\rho_F(D)$, considered in the context of polymer chemistry [6] is given by

$$\rho_F(D) = \frac{3}{(D+2)}, \quad D < 4, \tag{9}$$

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.04.073

which is in close agreement with the numerical result of [7] for $D = 3$. It is of interest here to compare the Flory formula (9) with the leading term on the right-hand side of Eq. (1) for the exponent of the critical isotherm.

To conclude, let us reiterate the interest for the future to study, probably numerically, a suitable fractal model to test and, if it then proves necessary, to refine Eq. (1) for $\delta(D)$. Secondly, it is also of major interest for later work in this area to return to Zhang's proposals for $D = 3$, and to establish the precise status of the two conjectures on which, as the author clearly points out, the basic statistical mechanics developed in [1] is entirely dependent.

Acknowledgements

One of us (D.J.K.) wishes to acknowledge financial support from the Welch Foundation. The other (N.H.M.) wishes to thank Professors W. Seitz and D.J. Klein for generous hospitality during a visit to Texas A&M at Galveston. Finally, thanks are also due to Professors P.M. Echenique and A. Rubio for their support during a visit to DIPC, San Sebastian, where N.H.M.'s contribution to this study was brought to final fruition.

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