ON CRUM'S PROBLEM

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In this article I give a solution of the following problem of M. Crum.

What is the maximum number of convex polyhedra, non-overlapping and such that any pair of them have a common boundary of positive area?

The answer to the similar plane problem is "four" and it was expected that a finite, rather small number, ten or twelve, would be the answer to the above problem. I shall show that actually the answer is "infinity".

Take rectangular coordinate axes and in the vertical plane XOZ draw a polygonal line $A_0 A_1 A_2 \ldots$, from $A_0(1, 0, 0)$, above OX, convex downwards, of total length less than $\frac{1}{4}$ and such that the angle between the directions OX and $A_n A_{n+1}$ is greater than $\frac{3}{4}\pi$ for any n.

Now take a sequence of positive numbers $\{\delta_n\}$ such that $\sum_1^\infty \delta_n < \frac{1}{4}$. Join the point $B_0(0, 1, 0)$ to A_0 and take the point D_1 on $B_0 A_0$ such that $B_0 D_1 = \delta_1$; join D_1 to A_1 and take the point D_2 on $D_1 A_1$ such that $D_1 D_2 = \delta_2$; then join D_2 to A_2 and take the point D_3 on $D_2 A_2$ such that $D_2 D_3 = \delta_3$, and so on. Denote by r_0 the plane XOY and by r_n, for $n = 1, 2, 3, \ldots$, the plane through A_{n-1}, D_n, A_n. Let B_n and C_n be respectively the points of intersection of r_n with OY and OZ. It is easy to see that every B_n is on the positive half of the Y-axis.

Denote by S_{k+1}, $k = 0, 1, 2, \ldots$, the polyhedron consisting of the points of the first octant that are not below any one of the planes $r_0, r_1, \ldots, r_k$ and not above the plane r_{k+1}.

If we denote by x^+, y^+ the half-spaces $x \geqslant 0$, $y \geqslant 0$, by r_n^+ the half-space of points on and above r_n, and by r_n^- the half-space of points on or below r_n, then $S_{k+1} = x^+ y^+ r_0^+ r_1^+ \ldots r_k^+ r_{k+1}^-$. Being the intersection of half-spaces, S_{k+1} is a convex polyhedron. By a direct inspection we see that the triangle $D_{k+1} A_{k+1} C_{k+1}$ belongs to the common boundary of S_{k+1} and S_{k+2}, and thus S_{k+1} *and* S_{k+2} *satisfy the required condition.* Denote the points of intersection of r_k, $k > 2$, with the lines $B_0 A_0$, $D_1 A_1$, $D_2 A_2$, ... by E_{0k}, E_{1k}, E_{2k}, ... respectively.

The triangle $A_1 D_1 A_0$ obviously forms a part of the surface of S_1. We also have

$$\Delta A_1 D_1 A_0 \subset r_0^+ r_1^+ r_2^+. \tag{1}$$

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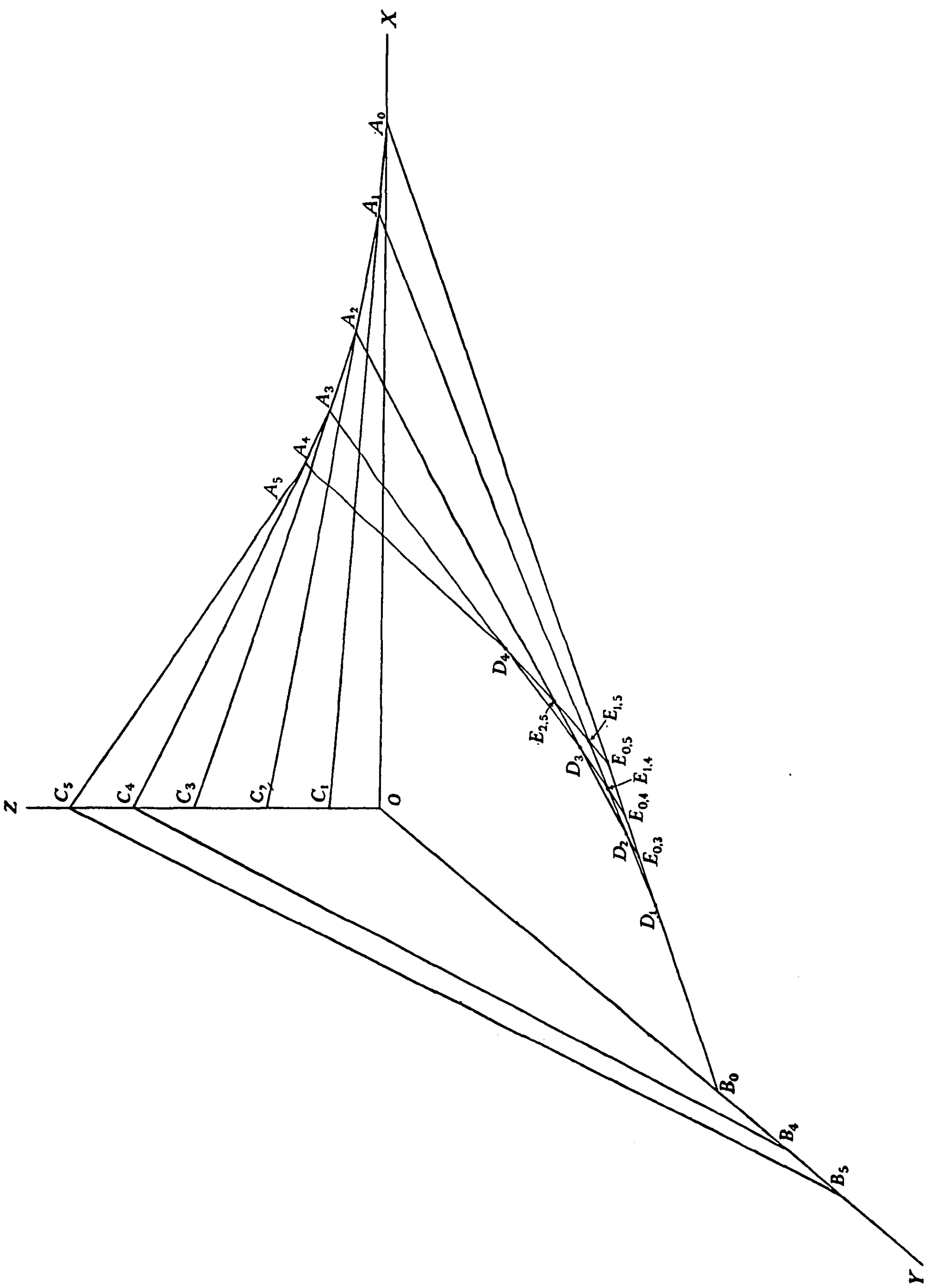
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The planes $r_3, r_4, \ldots, r_k, \ldots$ meet the triangle $A_1 D_1 A_0$ in the lines $D_2 E_{03}$, $E_{14} E_{04}, \ldots, E_{1k} E_{0k}, \ldots$ respectively, and the part of $A_1 D_1 A_0$ to the left of $E_{1k} E_{0k}$ is in r_k^-, and the one to the right in r_k^+; whence

$$D_2 D_1 E_{03} \subset r_3^-, \quad E_{14} D_2 E_{03} E_{04} \subset r_3^+ r_4^-, \quad E_{15} E_{14} E_{04} E_{05} \subset r_3^+ r_4^+ r_5^-, \quad \ldots,$$

and, by (1),

$$D_2 D_1 E_{03} \subset S_3, \quad E_{14} D_2 E_{03} E_{04} \subset S_4, \quad E_{15} E_{14} E_{04} E_{05} \subset S_5, \quad \ldots.$$

Thus S_1 has a common boundary of positive measure with any other S_k.

By considering the triangles $A_2 D_2 A_1$, $A_3 D_3 A_2$, ... we shall come to a similar conclusion with respect to S_2, S_3, ...

Thus any pair of polyhedra of the infinite sequence $\{S_k\}$ satisfy the required conditions.

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A SEQUENCE OF POLYHEDRA HAVING INTERSECTIONS OF SPECIFIED DIMENSIONS

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1. M. Crum proposed the following problem. What is the maximum number of non-overlapping convex polyhedra in 3-space which have the property that any two have a two-dimensional intersection? Besicovitch‡ proved that the answer is infinity, by constructing an infinite sequence of non-overlapping convex polyhedra any two of which have a two-dimensional intersection. In the present note I generalize this result by constructing a sequence of polyhedra $S_1, S_2, \ldots$ in n-space which have the property that, if $1 \leqslant k \leqslant \frac{1}{2}(n+1)$, any k of the S_m have an $(n-k+1)$-dimensional intersection. It is known that for $n=1$ and for $n=2$ the number $\frac{1}{2}(n+1)$ cannot be replaced by $\frac{1}{2}(n+1)+1$. All S_μ of our construction will lie in a fixed cube of side 2^{n+2}, and S_m, being the intersection of $2n+m+1$ half-spaces, will be a convex polyhedron.

THEOREM. *Let n be a positive integer. Let*

$$(1) \qquad 0 < t_0 < t_1 < \ldots; \quad t_\mu < 1 \quad (\mu \geqslant 0).$$

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‡ A. S. Besicovitch, *Journal London Math. Soc.*, 22 (1947), 285–287.