Biot-Savart Law

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1. Divergence and Curl

Let $v = \left(v^1, v^2, v^3\right)^T \in R^3$, we define

- **Divergence** $\nabla . v = \sum_{i=1}^{3} \partial_i v_i$;
- **Curl**

$$curl\ v = \begin{vmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= \left(\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1\right)$$
$$= \left(\omega_1, \omega_2, \omega_3\right) = \omega;$$

Also, the curl may be written as a matrix

$$\omega = \left(\omega_{ij}\right) = \left(\partial_i u_j - \partial_j u_i\right),$$

thus

$$\omega_1 = \omega_{23}, \omega_2 = \omega_{32}, \omega_3 = \omega_{12},$$

i.e.

$$\omega_i = \omega_{jk}$$

if $\{i, j, k\} = \{1, 2, 3\}$ and $[i, j, k]$ are of even sign.

2. Decomposition of a matrix

Let $A = (a_{ij}) \in R^{3\times3}$, then define the **deformation part** of A, $D(A)$ as

$$(D(A))_{ij} = \frac{a_{ij} + a_{ji}}{2}.$$

Thus for $v \in R^3$, $\nabla v = (\partial_i v_j) \in R^{3\times3}$, we have

$$\nabla v = (\partial_i v_j)$$
$$= \left(\frac{\partial_i v_j + \partial_j v_i}{2} + \frac{\partial_i v_j - \partial_j v_i}{2}\right) = \frac{D(A) + \omega}{2}.$$

3. Biot-Savart Law

Property For $v \in R^3$,

$$\Delta v = \nabla div\ v - curl\ curl\ v.$$

Proof This is just direct computation. Indeed,

$$\Delta v_i = \sum_{k=1}^{3} \partial_{kk}^2 v_i$$
$$= \sum_{k\neq i} \partial_k (\partial_k v_i) + \partial_i (\partial_i v_i)$$
$$= \sum_{k\neq i} \partial_k (\partial_k v_i) + \partial_i \left(div\ v - \sum_{k\neq i} \partial_k v_k\right)$$
$$= \partial_i div\ v + \sum_{k\neq i} \partial_k (\partial_k v_i - \partial_i v_k)$$
$$= \partial_i div\ v + \sum_{k=1}^{3} \partial_k \omega_{ki}.$$

Now the property follows, since for example,

$$\sum_{k=1}^{3} \partial_k \omega_{k1} = \partial_2 \omega_{21} + \partial_3 \omega_{31}$$
$$= -\partial_2 \omega_3 + \partial_3 \omega_2 = -(curl\ \omega)_1 .$$

Theorem (Biot-Savart Law) Let $v \in C_c^{\infty}(R^3)$, $\nabla . v = 0$, then

$$v(x) = -\frac{1}{4\pi} \int_{R^3} \frac{(x - y) \times \omega(y)}{|x - y|^3} dy .$$

Proof From

$$\Delta v_i = \sum_{k=1}^{3} \partial_k \omega_{ki}$$

and well-known Green's tensor for Δ in R^3 (c.f. GT),

$$v_i(x) = \frac{1}{4\pi} \int_{R^3} \frac{1}{|x - y|} \sum_{k=1}^{3} \partial_k \omega_{ki}(y)\ dy$$
$$= -\frac{1}{4\pi} \int_{R^3} \partial_k \frac{1}{|x - y|} \sum_{k=1}^{3} \omega_{ki}(y)\ dy$$
$$= \frac{1}{4\pi} \sum_{k=1}^{3} \int_{R^3} \frac{x_k - y_k}{|x - y|^3} \omega_{ki}(y)\, dy$$

Now we calculate for $v_1(x)$, while $v_2(x)$, $v_3(x)$ are similar.

$$
\begin{aligned}
v_1(x) &= \frac{1}{4\pi} \sum_{k=1}^{3} \int_{R^3} \frac{x_k - y_k}{|x-y|^3} \omega_{k1}(y)\, dy \\
&= \frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \begin{bmatrix} (x_2 - y_2)\,\omega_{21}(y) \\ +\,(x_3 - y_3)\,\omega_{31}(y) \end{bmatrix} dy \\
&= -\frac{1}{4\pi} \int_{R^3} \frac{1}{|x-y|} \begin{bmatrix} (x_2 - y_2)\,\omega_3 \\ -\,(x_3 - y_3)\,\omega_2 \end{bmatrix} dy \\
&= -\frac{1}{4\pi} \int_{R^3} \frac{[(x-y)\times\omega]_1}{|x-y|}\, dy.
\end{aligned}
$$

Remark 1° Of course one may extend the $C_c^\infty(R^3)$ class to general Sobolev classes.

2° From the formula

$$\Delta v = \nabla(div\ v) + curl\ \omega$$

We see readily that for $p \in (1, \infty)$,

$$\|\nabla v\|_{L^p} \leq C\left[\|div\ v\|_{L^p} + \|curl\ v\|_{L^p}\right]$$

from the standard L^p theory of elliptic operators.

Also,

$$
\begin{aligned}
&\|\nabla v\|_{L^p(B_R)} \\
&\leq C\left[\|div\ v\|_{L^p(B_R)} + \|curl\ v\|_{L^p(B_{2R})} + \|u\|_{L^p(B_{2R})}\right]
\end{aligned}
$$

from standard cut-off function technique.