A Note for P309 Smoller---Front Shock

I write this note to review and let me be familiar with shock theory.

Recall A k-shock $(k = 1, 2, \cdots, n)$ for the conservation laws

$$u_t + f(u)_x = 0, \; t > 0, x \in R^n.$$

is a hypersurface S with speed s where u is discontinuous through S, and

$$\begin{cases} \lambda_k(u_r) < s < \lambda_{k+1}(u_r), \\ \lambda_{k-1}(u_l) < s < \lambda_k(u_l). \end{cases}$$

Here

- (u_l, u_r) are the values of u on the left and right side of S, respectively;
- $\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$ are the eigenvalues of $df(u)$.

Now, we restrict ourselves to the p-sytems:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases} \qquad \text{(PE)}$$

where

- $v = \dfrac{1}{\rho}$ is the specific volume;

- u is the velocity;
- P is the pressure, with $p' < 0$, $p'' > 0$.

Written as a system of conservation laws, (PE) has the form

$$U_t + F(U)_x = 0,$$

Where

$$U = \begin{pmatrix} v \\ u \end{pmatrix}, \quad F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}.$$

Since the eigenvalues of $dF(U)$:

$$\lambda_1 = -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} = \lambda_2,$$

are real and distinct, the system (PE) is hyperbolic.

The 2-shock of (PE) is then such as

$$\begin{cases} \lambda_2(U_r) < s, \\ \lambda_1(U_l) < s < \lambda_2(U_l). \end{cases} \qquad (*)$$

Now the problem states:

Given a state $U_l = (v_l, u_l)$, find the possible state $U_r = (v_r, u_r)$ so that U_r is connected to U_l by a 2-shock on the right.

We do this just by the Rankine-Hugoniot-like conditions:

$$\begin{cases} s(v_r - v_l) = -(u_r - u_l), \\ s(u_r - u_l) = p(v_r) - p(v_l). \end{cases}$$

Eliminating s from the these equations we obtain

$$u_r - u_l = \pm\sqrt{(p(v_l) - p(v_r))(v_r - v_l)}. \quad (**)$$

So our next goal is to determine the sign in (*).

- (*) implies that

$$\sqrt{-p'(v_r)} < \sqrt{-p'(v_l)},$$

thus

$$v_r < v_l.$$

- Then

$$\left.\begin{matrix} (**)_1 \\ v_r < v_l \\ s > 0 \end{matrix}\right\} \Rightarrow u_l < u_r.$$

- The sign thus is $+$.

Now the front-shock (or 2-shock) has the formula:

$$S_2 : u_r - u_l = \sqrt{(p(v_l) - p(v_r))(v_r - v_l)} = s_1(v_r; U_l), \ v_l < v_r.$$

As pointed out before, S_2 is star-like w.r.t. U_l. And the picture of the 2-shock is easily depicted.

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