Note for P311 Smoller---Second order contact

The first-order characteristic family:

- **1-shock curve**:

$$u - u_1 = -\sqrt{(v - v_1)\left(p(v_1) - p(v)\right)}$$
$$= s_1(v; U_1), \ v_1 > v.$$

- **1-rarefaction curve**:

$$u - u_1 = \int_{v_1}^{v} \sqrt{-p'(y)}dy, \ v > v_1.$$

Property s_1 and r_1 have second order contact at U_1.

Proof Let

$$\begin{cases} \omega = v - v_1 < 0, \\ f(\omega) = p(\omega + v_1) - p(v_1) = p(v) - p(v_1). \end{cases}$$

Then

$$f(0) = 0, \ f'(\omega) = p'(v) < 0, \ f''(\omega) = p''(v) > 0.$$

And

$$s_1 = -\sqrt{-\omega f(\omega)} = s(\omega), \quad (*)$$

$$s^2(\omega) = -\omega f(\omega). \quad (**)$$

- Differentiating (**) gives

$$2ss' = -f(\omega) - \omega f'(\omega), \quad (***)$$

thus

$$s' = \frac{-f(\omega) - \omega f'(\omega)}{2s}$$

$$= \frac{-f(\omega) - \omega f'(\omega)}{-2\sqrt{-\omega f(\omega)}} \qquad (****)$$

$$= \frac{\frac{f(\omega)}{\omega} + f'(\omega)}{-2\sqrt{\frac{f(\omega)}{-\omega}}}.$$

Hence

$$s_1'(v_1 - 0) = s'(0 - 0)$$

$$= \frac{f'(0) + f'(0)}{-2\sqrt{-f'(0)}} = \frac{-f'(0)}{\sqrt{-f'(0)}} = \sqrt{-f'(0)}$$

$$= \sqrt{-p'(v_1)}$$

$$= r_1'(v_1 + 0).$$

- **Now differentiating (***), and using (*), (****), we have**

$$2[s']^2 + 2ss'' = -2f'(\omega) - \omega f''(\omega),$$

and

$$s'' = \frac{-2f'(\omega) - \omega f''(\omega) - 2[s']^2}{2s}$$

$$= \frac{-2f'(\omega) - \omega f''(\omega) - 2\left[\frac{f(\omega) + \omega f'(\omega)}{2s}\right]^2}{2s}$$

$$= \frac{2s^2[-2f'(\omega) - \omega f''(\omega)] - [f(\omega) + \omega f'(\omega)]^2}{4s^3}$$

$$= \frac{4\omega f(\omega) f'(\omega) + 2\omega^2 f(\omega) f''(\omega) - [f(\omega) + \omega f'(\omega)]^2}{4[-\omega f(\omega)]^{\frac{3}{2}}}$$

$$= \frac{2\omega^2 f(\omega) f''(\omega) - [f(\omega) - \omega f'(\omega)]^2}{-4[-\omega f(\omega)]^{\frac{3}{2}}}.$$

While

$$
\begin{aligned}
&\lim_{\omega\to 0-} \frac{\left[f(\omega) - \omega f'(\omega)\right]^2}{\omega^3} \\
&= \lim_{\omega\to 0-} \frac{2\left[f(\omega) - \omega f'(\omega)\right]\left[-\omega f''(\omega)\right]}{3\omega^2} \\
&= \lim_{\omega\to 0-} \frac{-2}{3}\left[\frac{f(\omega)}{\omega} - f'(\omega)\right] f''(\omega) \\
&= 0.
\end{aligned}
$$

We finally obtain

$$
\begin{aligned}
s_1''(v_1 - 0) &= s''(0 - 0) \\
&= \lim_{\omega\to 0-} \frac{-2\dfrac{f(\omega)}{\omega} f''(\omega) + \dfrac{\left[f(\omega) - \omega f'(\omega)\right]^2}{\omega^3}}{-4\left[-\dfrac{f(\omega)}{\omega}\right]^{\frac{3}{2}}} \\
&= \frac{-f'(0) f''(0)}{-4\left[-f'(0)\right]^{\frac{3}{2}}} \\
&= -\frac{f''(0)}{2\sqrt{-f'(0)}} \\
&= -\frac{p''(v_1)}{2\sqrt{-p'(v_1)}} \\
&= r_1''(v_1 + 0).
\end{aligned}
$$

The proof is complete.

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