Chapter 3 Random Variables and Processes

3.1. Random Variables

Consider an experiment with outcomes defined by a certain sample space. The rule or functional relationship that maps each point in this sample space into a real number is called a random variable. Random variables are designated by capital letters (e.g., *X*, *Y*, …), and a particular value of a random variable is denoted by a lowercase letter (e.g., *x*, *y*, …).

The Cumulative Distribution Function (*cdf*) associated with the random variable X is denoted as $F_X(x)$ and is interpreted as the total probability that the random variable X is less than or equal to the value x. More precisely,

$$F_X(x) = Pr\{X \le x\} \tag{3.1}$$

The probability that the random variable X is in the interval (x_1, x_2) is then given by

$$F_X(x_2) - F_X(x_1) = Pr\{x_1 \le X \le x_2\} \tag{3.2}$$

The probability that a random variable X has values in the interval (x_1, x_2) is

$$F_X(x_2) - F_X(x_1) = Pr\{x_1 \le X \le x_2\} = \int_{x_1}^{x_2} f_X(x)dx \tag{3.3}$$

It is often practical to describe a random variable by the derivative of its *cdf*, which is called the Probability Density Function *(pdf)*. The *pdf* of the random variable X is

$$f_X(x) = \frac{d}{dx}F_X(x) \tag{3.4}$$

or, equivalently,

$$F_X(x) = Pr\{X \le x\} = \int_{-\infty}^{x} f_X(\lambda) d\lambda \tag{3.5}$$

The *cdf* has the following properties:

$$\begin{aligned} 0 \le F_X(x) \le 1 \\ F_X(-\infty) = 0 \\ F_X(\infty) = 1 \\ F_X(x_1) \le F_X(x_2) \Leftrightarrow x_1 \le x_2 \end{aligned} \tag{3.6}$$

Define the nth moment for the random variable X as

$$E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x) dx \tag{3.7}$$

The first moment, $E[X]$, is called the mean value, while the second moment, $E[X^2]$, is called the mean squared value. When the random variable X represents an electrical signal across a 1Ω resistor, then $E[X]$ is the DC component, and $E[X^2]$ is the total average power.

The nth central moment is defined as

$$E[(X - \bar{X})^n] = \overline{(X - \bar{X})^n} = \int_{-\infty}^{\infty} (x - \bar{x})^n f_X(x) dx \tag{3.8}$$

and, thus, the first central moment is zero. The second central moment is called the variance and is denoted by the symbol σ_X^2,

$$\sigma_X^2 = \overline{(X - \bar{X})^2} \tag{3.9}$$

In practice, the random nature of an electrical signal may need to be described by more than one random variable. In this case, the joint *cdf* and *pdf* functions need to be considered. The joint *cdf* and *pdf* for the two random variables X and Y are, respectively, defined by

$$F_{XY}(x, y) = Pr\{X \le x; Y \le y\} \tag{3.10}$$

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y) \tag{3.11}$$

The marginal *cdf*s are obtained as follows:

$$F_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{UV}(u, v) du dv = F_{XY}(x, \infty)$$
$$F_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{UV}(u, v) dv du = F_{XY}(\infty, y) \tag{3.12}$$

If the two random variables are statistically independent, then the joint *cdf*s and *pdf*s are, respectively, given by

$$F_{XY}(x, y) = F_X(x) F_Y(y) \tag{3.13}$$

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{3.14}$$

Let us now consider a case when the two random variables X and Y are mapped into two new variables U and V through some transformations T_1 and T_2 defined by

$$U = T_1(X, Y) \quad ; \quad V = T_2(X, Y) \tag{3.15}$$

The joint *pdf*, $f_{UV}(u, v)$, may be computed based on the invariance of probability under the transformation. One must first compute the matrix of derivatives; then the new joint *pdf* is computed as

$$f_{UV}(u, v) = f_{XY}(x, y) |J| \tag{3.16}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{3.17}$$

where the determinant of the matrix of derivatives $|J|$ is called the Jacobian. The characteristic function for the random variable X is defined as

$$C_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \tag{3.18}$$

The characteristic function can be used to compute the *pdf* for a sum of independent random variables. More precisely, let the random variable Y be

$$Y = X_1 + X_2 + \ldots + X_N \tag{3.19}$$

where $\{X_i \; ; \; i = 1, \ldots, N\}$ is a set of independent random variables. It can be shown that

$$C_Y(\omega) = C_{X_1}(\omega)C_{X_2}(\omega)\dots C_{X_N}(\omega) \tag{3.20}$$

and the *pdf* $f_Y(y)$ is computed as the inverse Fourier transform of $C_Y(\omega)$ (with the sign of y reversed):

$$f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_Y(\omega)e^{-j\omega y}d\omega \tag{3.21}$$

The characteristic function may also be used to compute the nth moment for the random variable X as

$$E[X^n] = (-j)^n \frac{d^n}{d\omega^n}C_X(\omega)\bigg|_{\omega = 0} \tag{3.22}$$

3.2. Multivariate Gaussian Random Vector

Consider a joint probability for m random variables, $X_1, X_2, \dots, X_m$. These variables can be represented as components of an $m \times 1$ random column vector, $\mathbf{X}$. More precisely,

$$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \dots & X_m \end{bmatrix}^t \tag{3.23}$$

where the superscript t indicates the transpose operation. The joint *pdf* for the vector $\mathbf{X}$ is

$$f_X(\underline{x}) = f_{X_1, X_2, \dots, X_m}(x_1, x_2, \dots, x_m) \tag{3.24}$$

The mean vector is defined as

$$\mu_X = \begin{bmatrix} E[X_1] & E[X_2] & \dots & E[X_m] \end{bmatrix}^t \tag{3.25}$$

and the covariance is an $m \times m$ matrix given by

$$\boldsymbol{C}_X = E[\mathbf{X}\ \mathbf{X}^t] - \mu_X\ \mu_X^t \tag{3.26}$$

Note that if the elements of the vector $\mathbf{X}$ are independent, then the covariance matrix is a diagonal matrix.

A random vector $\mathbf{X}$ is multivariate Gaussian if its *pdf* is of the form

$$f_X(\underline{x}) = \frac{1}{\sqrt{(2\pi)^m|\boldsymbol{C}_X|}}\exp\left(-\frac{1}{2}(\boldsymbol{x} - \mu_X)^t\boldsymbol{C}_X^{-1}(\boldsymbol{x} - \mu_X)\right) \tag{3.27}$$

where μ_x is the mean vector, $\boldsymbol{C}_x$ is the covariance matrix, $\boldsymbol{C}_x^{-1}$ is inverse of the covariance matrix and $|\boldsymbol{C}_x|$ is its determinant, and $\mathbf{X}$ is of dimension m. If $\boldsymbol{A}$[1] is a $k \times m$ matrix of rank k, then the random vector $\mathbf{Y} = \boldsymbol{A}\mathbf{X}$ is a k-variate Gaussian vector with

$$\mu_Y = \boldsymbol{A}\mu_X \tag{3.28}$$

$$\boldsymbol{C}_Y = \boldsymbol{A}\Lambda_X \boldsymbol{A}^t \tag{3.29}$$

The characteristic function for a multivariate Gaussian *pdf* is defined by

$$C_X = E[\exp\{j(\omega_1 X_1 + \omega_2 X_2 + \ldots + \omega_m X_m)\}] = \tag{3.30}$$

$$\exp\left\{j\mu_X^t\omega - \frac{1}{2}\omega^t \boldsymbol{C}_X \omega\right\}$$

Then the moments for the joint distribution can be obtained by partial differentiation. For example,

$$E[X_1 X_2 X_3] = \frac{\partial^3}{\partial\omega_1 \partial\omega_2 \partial\omega_3} C_X(\omega_1, \omega_2, \omega_3) \qquad at \quad \omega = \mathbf{0} \tag{3.31}$$

Example:

The vector $\mathbf{X}$ is a 4-variate Gaussian with

$$\mu_X = \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}^t \; and \; \boldsymbol{C}_X = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 3 \end{bmatrix}$$

Define

$$\mathbf{X}_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \mathbf{X}_2 = \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}$$

Find the distribution of $\mathbf{X}_1$ and the distribution of

1. Note that matrices are denoted by italicized upper case bold face letters, while vectors are denoted by lower and upper regular (not italicized) letters.

$$\mathbf{Y} = \begin{bmatrix} 2\mathbf{X}_1 \\ \mathbf{X}_1 + 2\mathbf{X}_2 \\ \mathbf{X}_3 + \mathbf{X}_4 \end{bmatrix}$$

Solution:

$\mathbf{X}_1$ has a bivariate Gaussian distribution with

$$\mu_{X_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{C}_{X_1} = \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}$$

The vector $\mathbf{Y}$ can be expressed as

$$\mathbf{Y} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \mathbf{X}_3 \\ \mathbf{X}_4 \end{bmatrix} = \boldsymbol{A}\mathbf{X}$$

It follows that

$$\mu_Y = \boldsymbol{A}\mu_X = \begin{bmatrix} 4 & 4 & 1 \end{bmatrix}^t$$

$$\boldsymbol{C}_Y = \boldsymbol{A}\boldsymbol{C}_X\ \boldsymbol{A}^t = \begin{bmatrix} 24 & 24 & 6 \\ 24 & 34 & 13 \\ 6 & 13 & 13 \end{bmatrix}$$

A special case of Eq. (3.29) is when the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} a_1 a_2 \ \dots \ a_m \end{bmatrix} \tag{3.32}$$

It follows that $\mathbf{Y} = \boldsymbol{A}\mathbf{X}$ is a sum of random variables X_i, that is

$$Y = \sum_{i=1}^{m} a_i\ X_i \tag{3.33}$$

The finding in Eq. (3.33) leads to the conclusion that the linear sum of Gaussian variables is also a Gaussian variable with mean and variance given by

$$\overline{Y} = a_1\overline{X}_1 + a_2\overline{X}_2 + \dots + a_m\overline{X}_m \tag{3.34}$$

$$\sigma_Y^2 = E[(X-\bar{X})^2] = \\ E[a_1(X_1-\bar{X}_1) + a_2(X_2-\bar{X}_2) + \ldots + a_m(X_m-\bar{X}_m)] \quad (3.35)$$

and if the variables X_i are independent then Eq.(3.35) reduces to

$$\sigma_Y^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + \ldots + a_m^2\sigma_{X_m}^2 \quad (3.36)$$

finally, in this case, the probability density function $f_Y(y)$ is given by (which can also be derived from Eq. (3.20))

$$f_Y(y) = f_{X_1}(x_1) \otimes f_{X_2}(x_2) \otimes \ldots \otimes f_{X_m}(x_m) \quad (3.37)$$

where $\otimes$ indicates convolution.

3.2.1. Complex Multivariate Gaussian Random Vector

Consider the complex vector random variable

$$\tilde{\mathbf{X}} = \mathbf{X}_I + j\mathbf{X}_Q \quad (3.38)$$

where $\mathbf{X}_I$ and $\mathbf{X}_Q$ are real random multivariate Gaussian random vectors. The joint *pdf* for the complex random vector $\tilde{\mathbf{X}}$ is computed from the joint *pdf* of the two real vectors. The mean for the vector $\tilde{\mathbf{X}}$ is

$$E[\tilde{\mathbf{X}}] = E[\mathbf{X}_I] + jE[\mathbf{X}_Q] \quad (3.39)$$

The covariance matrix is also defined by

$$\tilde{\mathbf{C}} = E[(\tilde{\mathbf{X}} - E[\tilde{\mathbf{X}}])(\tilde{\mathbf{X}} - E[\tilde{\mathbf{X}}])^{\dagger}] \quad (3.40)$$

where the operator $\dagger$ indicates complex conjugate transpose.

The *pdf* for the vector $\tilde{\mathbf{X}}$ is

$$f_{\tilde{X}}(\tilde{\mathbf{x}}) = \frac{\exp[-(\tilde{\mathbf{x}} - E[\mathbf{x}])^{\dagger}\tilde{\mathbf{C}}^{-1}(\tilde{\mathbf{x}} - E[\mathbf{x}])]}{\pi^N|\tilde{\mathbf{C}}|} \quad (3.41)$$

with the following three conditions holding true

$$E[(\mathbf{X}_{I_i} - E[\mathbf{X}_{I_i}])(\mathbf{X}_{Q_i} - E[\mathbf{X}_{Q_i}])^{\dagger}] = \mathbf{0} \quad (3.42)$$

$$E[(\mathbf{X}_{I_i} - E[\mathbf{X}_{I_i}])(\mathbf{X}_{I_k} - E[\mathbf{X}_{I_k}])^{\dagger}] = \\ E[(\mathbf{X}_{Q_i} - E[\mathbf{X}_{Q_i}])(\mathbf{X}_{Q_k} - E[\mathbf{X}_{Q_k}])^{\dagger}] \;;all \;\; i, k \quad (3.43)$$

$$E[(\mathbf{X}_{I_i} - E[\mathbf{X}_{I_i}])(\mathbf{X}_{Q_k} - E[\mathbf{X}_{Q_k}])^\dagger] = \\ -E[(\mathbf{X}_{Q_i} - E[\mathbf{X}_{Q_i}])(\mathbf{X}_{I_k} - E[\mathbf{X}_{I_k}])^\dagger] \quad ;all \quad i \neq k \tag{3.44}$$

3.3. Rayleigh Random Variables

Let X_I and X_Q be zero mean independent Gaussian random variables with zero mean and variance σ^2. Define two new random variables R and Φ as

$$X_I = R\cos\Phi \\ X_Q = R\sin\Phi \tag{3.45}$$

The joint *pdf* of the two random variables $X_I;X_Q$ is

$$f_{X_I X_Q}(x_I, x_Q) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_I^2 + x_Q^2}{2\sigma^2}\right) = \\ \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(r\cos\varphi)^2 + (r\sin\varphi)^2}{2\sigma^2}\right) \tag{3.46}$$

The joint *pdf* for the two random variables $R;\Phi$ is given by

$$f_{R\Phi}(r, \varphi) = f_{X_I X_Q}(x_I, x_Q)|\boldsymbol{J}| \tag{3.47}$$

where $[\boldsymbol{J}]$ is a matrix of derivatives defined by

$$[\boldsymbol{J}] = \begin{bmatrix} \dfrac{\partial x_I}{\partial r} & \dfrac{\partial x_I}{\partial \varphi} \\ \dfrac{\partial x_Q}{\partial r} & \dfrac{\partial x_Q}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix} \tag{3.48}$$

The determinant of the matrix of derivatives is called the Jacobian, and in this case it is equal to

$$|\boldsymbol{J}| = r \tag{3.49}$$

Substituting Eqs. (3.46) and (3.49) into Eq. (3.47) and collecting terms yield

$$f_{R\Phi}(r, \varphi) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{(r\cos\varphi)^2 + (r\sin\varphi)^2}{2\sigma^2}\right) = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{3.50}$$

The *pdf* for R alone is obtained by integrating Eq. (3.50) over φ

$$f_R(r) = \int_0^{2\pi} f_{R\Phi}(r, \varphi) d\varphi = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \frac{1}{2\pi} \int_0^{2\pi} d\varphi \tag{3.51}$$

where the integral inside Eq. (3.51) is equal to 2π; thus,

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad ; r \geq 0 \tag{3.52}$$

The *pdf* described in Eq. (3.52) is referred to as a Rayleigh probability density function.

The density function for the random variable Φ is obtained from

$$f_\Phi(\varphi) = \int_0^r f(r, \varphi) \; dr \tag{3.53}$$

substituting Eq. (3.50) into Eq. (3.53) and performing integration by parts yields

$$f_\Phi(\varphi) = \frac{1}{2\pi} \quad ; \; 0 < \varphi < 2\pi \tag{3.54}$$

which is a uniform probability density function.

3.4. The Chi-Square Random Variables

3.4.1. Central Chi-Square Random Variable with N Degrees of Freedom

Let the random variables $\{X_1, X_2, \ldots, X_N\}$ be zero mean, statistically independent Gaussian random variable with unity variance. The variable

$$\chi_N^2 = \sum_{i=1}^{N} X_i^2 \tag{3.55}$$

is called a central chi-square random variable with N degrees of freedom. The chi-square *pdf* is

$$f_{\chi_N^2}(x) = \begin{cases} \dfrac{x^{(N-2)/2} \; e^{(-x/2)}}{2^{N/2} \; \Gamma(N/2)} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.56}$$

where the Gamma function is define as

$$\Gamma(n) = \int_0^{\infty} \lambda^{n-1} e^{-\lambda} \, d\lambda \quad ; \; n > 0 \tag{3.57}$$

with the following recursion

$$\Gamma(n+1) = n\Gamma(n) \tag{3.58}$$

and

$$\Gamma(n+1) = n! \qquad ; \; n = 0, 1, 2, \ldots, and \;\; 0! = 1 \tag{3.59}$$

The mean and variance for the central chi-square are, respectively given by

$$E[\chi_N^2] = N \tag{3.60}$$

$$\sigma_{\chi_N^2} = 2N \tag{3.61}$$

Hence, the degrees of freedom N is the ratio of twice the squared mean to the variance

$$N = (2E^2[\chi_N^2]) / \sigma_{\chi_N^2} \tag{3.62}$$

3.4.2. Noncentral Chi-Square Random Variable with N Degrees of Freedom

In the general case, the chi-square random variable requires that the Gaussian random variables $\{X_1, X_2, \ldots, X_N\}$ do not have zero means. Define a multivariate random variable $\mathbf{Y}$ such that

$$Y_i = X_i + \mu_{X_i} \quad ; i = 1, 2, \ldots, N \tag{3.63}$$

Consider the random variable

$$\chi'^2_N = \sum_{i=1}^{N} Y_i^2 = \sum_{i=1}^{N} (X_i + \mu_{X_i})^2 \tag{3.64}$$

the variable χ'^2_N is called the noncentral chi-square random variable with N degrees of freedom and with a noncentral parameter λ, where

$$\lambda = \sum_{i=1}^{N} \mu_{X_i}^2 = \sum_{i=1}^{N} E^2[Y_i] \tag{3.65}$$

The noncentral chi-square *pdf* is

$$f_{\chi'^2_N}(x) = \begin{cases} \left(\frac{1}{2}\right)\left(\frac{x}{\lambda}\right)^{(N-2)/4} e^{[-(x+\lambda)/2]I_{(N-2)/2}(\sqrt{\lambda x})} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.66}$$

where I is the modified Bessel function (or occasionally called the hyperbolic Bessel function) of the first kind; and the subscripts is referred to as its order.

3.5. Random Processes

A random variable X is by definition a mapping of all possible outcomes of a random experiment to numbers. When the random variable becomes a function of both the outcomes of the experiment time, it is called a random process and is denoted by $X(t)$. Thus, one can view a random process as an ensemble of time-domain functions that are the outcome of a certain random experiment, as compared with single real numbers in the case of a random variable.

Since the *cdf* and *pdf* of a random process are time dependent, we will denote them as $F_X(x;t)$ and $f_X(x;t)$, respectively. The nth moment for the random process $X(t)$ is

$$E[X^n(t)] = \int_{-\infty}^{\infty} x^n f_X(x;t)dx \tag{3.67}$$

A random process $X(t)$ is referred to as stationary to order one if all its statistical properties do not change with time. Consequently, $E[X(t)] = \overline{X}$, where $\overline{X}$ is a constant. A random process $X(t)$ is called stationary to order two (or wide-sense stationary) if

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t) \tag{3.68}$$

for all t_1, t_2 and Δt.

Define the statistical autocorrelation function for the random process $X(t)$ as

$$\Re_X(t_1, t_2) = E[X(t_1)X(t_2)] \tag{3.69}$$

The correlation $E[X(t_1)X(t_2)]$ is, in general, a function of (t_1, t_2). As a consequence of the wide-sense stationary definition, the autocorrelation function depends on the time difference $\tau = t_2 - t_1$, rather than on absolute time; and thus, for a wide-sense stationary process we have

$$\begin{aligned} E[X(t)] &= \bar{X} \\ \Re_X(\tau) &= E[X(t)X(t+\tau)] \end{aligned} \tag{3.70}$$

If the time average and time correlation functions are equal to the statistical average and statistical correlation functions, the random process is referred to as an ergodic random process. The following is true for an ergodic process:

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt = E[X(t)] = \bar{X} \tag{3.71}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)x(t+\tau)dt = \Re_X(\tau) \tag{3.72}$$

The covariance of two random processes $X(t)$ and $Y(t)$ is defined by

$$C_{XY}(t, t+\tau) = E[\{X(t) - E[X(t)]\}\{Y(t+\tau) - E[Y(t+\tau)]\}] \tag{3.73}$$

which can also be written as

$$C_{XY}(t, t+\tau) = \Re_{XY}(\tau) - \bar{X}\bar{Y} \tag{3.74}$$

3.6. Bandpass Gaussian Random Process

It is customary to define the bandpass Gaussian random process through its complex envelope as

$$\tilde{X}(t) = X_I(t) + jX_Q(t) \tag{3.75}$$

where both $X_I(t)$ and $X_Q(t)$ are lowpass Gaussian random processes with zero mean and variance σ^2. The *pdf* for a sample $\tilde{X}(t_0)$ of the complex envelope is the joint *pdf* for $X_I(t)$ and $X_Q(t)$. That is,

$$f_{\tilde{X}}(\tilde{x}(t_0)) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{x_I^2(t_0) + x_Q^2(t_0)}{2\sigma^2}\right] = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{|\tilde{x}(t_0)|^2}{2\sigma^2}\right] \tag{3.76}$$

Now, if both lowpass processes do not have zero mean and instead have a mean defined by

$$\mu(t) = \mu_I(t)\cos(2\pi f_0 t) + j\mu_Q(t)\sin(2\pi f_0 t) \tag{3.77}$$

the mean complex envelope is

$$\tilde{\mu}(t) = \mu_I(t) + j\mu_Q(t) \tag{3.78}$$

It follows that Eq. (3.76) can be rewritten as

$$f_{\tilde{X}}(\tilde{x}(t_0)) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{[x_I(t_0)-\mu_I(t_0)]^2+[x_Q(t_0)-\mu_Q(t_0)]^2}{2\sigma^2}\right] = \frac{1}{2\pi\sigma^2}\exp\left[\frac{\left|\tilde{x}(t_0)-\tilde{\mu}(t_0)\right|^2}{2\sigma^2}\right] \tag{3.79}$$

Consider a duration of the process than spans the interval $\{0, T_0\}$. Then this segment of the complex envelope of the random process can be represented using a complex random variable vector of at least $N = BT_0$ elements where B is the bandwidth of the process. Define

$$\tilde{X}_i = \tilde{X}\left(\frac{i}{B}\right) \quad ;i = 1, 2, \ldots, BT_0 \tag{3.80}$$

$$\tilde{\mathbf{X}}^\dagger = \left[\tilde{X}_1\ \tilde{X}_2\ \ldots\tilde{X}_{BT_0}\right] \tag{3.81}$$

By definition the covariance matrix $\tilde{\boldsymbol{C}}$ is

$$\tilde{\boldsymbol{C}} = E[(\tilde{\mathbf{X}}-\tilde{\mu})(\tilde{\mathbf{X}}-\tilde{\mu})^\dagger] = 2(\tilde{\boldsymbol{C}}_I + j\tilde{\boldsymbol{C}}_{IQ}) \tag{3.82}$$

where

$$\tilde{\boldsymbol{C}}_I = E[(\tilde{\mathbf{X}}_I-\tilde{\mu}_I)(\tilde{\mathbf{X}}_I-\tilde{\mu}_I)^\dagger] \tag{3.83}$$

$$\tilde{\boldsymbol{C}}_{IQ} = E[(\tilde{\mathbf{X}}_I-\tilde{\mu}_I)(\tilde{\mathbf{X}}_Q-\tilde{\mu}_Q)^\dagger] \tag{3.84}$$

Therefore, the *pdf* for the segment $\{\tilde{X}(t) \ ;0 < t < T_0\}$ is

$$f_{\tilde{X}}(\tilde{\mathbf{x}}) = \frac{\exp[-(\tilde{\mathbf{x}}-\tilde{\mu})^\dagger\tilde{\boldsymbol{C}}^{-1}(\tilde{\mathbf{x}}-\tilde{\mu})]}{\pi^N|\tilde{\boldsymbol{C}}|} \tag{3.85}$$

3.6.1. The Envelope of a Bandpass Gaussian Process

Consider the *pdf* of a segment of the envelope of a bandpass Gaussian random process. This process can expressed as

$$X(t) = X_I(t)\cos(2\pi f_0 t) - X_Q(t)\sin(2\pi f_0 t) \tag{3.86}$$

where $X_I(t)$ and $X_Q(t)$ are zero mean independent lowpass Gaussian processes. The envelope and phase are respectively denoted by $R(t)$ and $\Phi(t)$, where

$$R(t) = \sqrt{X_I(t)^2 + X_Q(t)^2} \tag{3.87}$$

and

$$\Phi(t) = \left[\tan\left(\frac{X_Q(t)}{X_I(t)}\right)\right]^{-1} \tag{3.88}$$

where

$$\begin{aligned} X_I(t) &= R(t)\cos(\Phi(t)) \\ X_Q(t) &= R(t)\sin(\Phi(t)) \end{aligned} \tag{3.89}$$

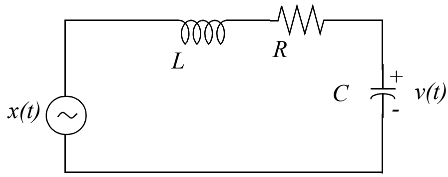
The two processes $R(t)$ and $\Phi(t)$ are also independent, and their respective *pdfs* were derived in Section 3.3 and were given in Eqs. (3.52) and (3.54), respectively.

Problems

3.1. Suppose you want to determine an unknown DC voltage v_{dc} in the presence of additive white Gaussian noise $n(t)$ of zero mean and variance σ_n^2. The measured signal is $x(t) = v_{dc} + n(t)$. An estimate of v_{dc} is computed by making three independent measurements of $x(t)$ and computing the arithmetic mean, $\underset{\sim}{v}_{dc} \approx (x_1 + x_2 + x_3)/3$. (a) Find the mean and variance of the random variable $\underset{\sim}{v}_{dc}$. (b) Does the estimate of v_{dc} get better by using ten measurements instead of three? Why?

3.2. Assume the X and Y miss distances of darts thrown at a bulls-eye dart board are Gaussian with zero mean and variance σ^2. (a) Determine the probability that a dart will fall between 0.8σ and 1.2σ. (b) Determine the radius of a circle about the bull's-eye that contains 80% of the darts thrown. (c) Consider a square with side s in the first quadrant of the board. Determine s so that the probability that a dart will fall within the square is 0.07.

3.3. (a) A random voltage $v(t)$ has an exponential distribution function $f_V(v) = a\exp(-av)$, where $(a > 0);(0 \le v < \infty)$. The expected value $E[V] = 0.5$. Determine $Pr\{V > 0.5\}$. Consider the network shown in figure below, where $x(t)$ is a random voltage with zero mean and autocorrelation function $\Re_x(\tau) = 1 + \exp(-a|t|)$. Find the power spectrum $S_x(\omega)$. What is the transfer function? Find the power spectrum $S_v(\omega)$.



3.4. Let $\bar{S}_X(\omega)$ be the PSD function for the stationary random process $X(t)$. Compute an expression for the PSD function of

$$Y(t) = X(t) - 2X(t-T).$$

3.5. Let X be a random variable with

$$f_X(x) = \begin{Bmatrix} \frac{1}{\sigma} t^3 e^{-t} & t \geq 0 \\ 0 & elsewhere \end{Bmatrix}$$

(a) Determine the characteristic function $C_X(\omega)$. (b) Using $C_X(\omega)$, validate that $f_X(x)$ is a proper *pdf.* (c) Use $C_X(\omega)$ to determine the first two moments of X. (d) Calculate the variance of X.

3.6. Let the random variable Z be written in terms of two other random variables X and Y as follows: $Z = X + 3Y$. Find the mean and variance for the new random variable in terms of the other two.

3.7. Suppose you have the following sequences of statistically independent Gaussian random variables with zero means and variances σ^2. if

$$X_1, X_2, \ldots, X_N \ ; \ X_i = A_i \cos\Theta_i \text{ and } Y_1, Y_2, \ldots, Y_N \ ; \ Y_i = A_i \sin\Theta_i .$$

Define $Z = \sum_{i=1}^{N} A_i^2$. Find an expression that Z exceeds a threshold value v_T.

3.8. Repeat the previous problem when two single delay line cancellers are cascaded to produce a double delay line canceller.Let $X(t)$ be a stationary random process, $E[X(t)] = 1$ and the autocorrelation $\Re_x(\tau) = 3 + \exp(-|\tau|)$. Define a new random variola Y as

$$Y = \int_0^2 x(t)dt$$

Compute $E[Y(t)]$ and σ_Y^2.

3.9. Consider the single delay line canceller in the figure below. The input $x(t)$ is a wide sense stationary random process with variance σ_x^2 and mean μ_x and a covariance matrix Λ. Find the mean and variance and the autocorrelation function of the output $y(t)$.