Chapter 4 The Matched Filter

4.1. The Matched Filter SNR

The topics of matched filtering and pulse compression (see Chapter 8) are central to almost all radar systems. In this chapter the focus is the matched filter. The unique characteristic of the matched filter is that it produces the maximum achievable instantaneous SNR at its output when a signal plus additive white noise is present at the input. Maximizing the SNR is key in all radar applications, as was described in Chapter 1 in the context of the radar equation and as will be discussed in Chapter 7 in the context of target detection.

Therefore, it is important to use a radar receiver which can be modeled as an LTI system that maximizes the signal's SNR at its output. For this purpose, the basic radar receiver of interest is often referred to as the matched filter receiver. The matched filter is an optimum filter in the sense of SNR because the SNR at its output is maximized at some delay t_0 that corresponds to the true target range R_0 (i.e., $t_0 = (2R_0)/c$). Figure 4.1 shows a simplified block diagram for the radar receiver of interest.

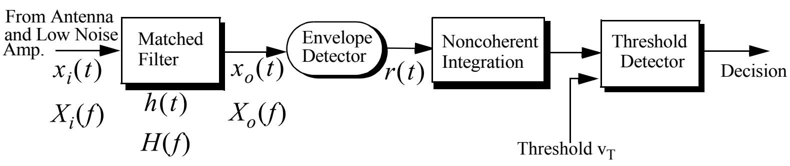

Figure 4.1. Simplified block diagram of the radar receiver

In order to derive the general expression for the transfer function and the impulse response of this optimum filter, adopt the following notation:

$h(t)$ is the optimum filter impulse response
$H(f)$ is the optimum filter transfer function
$x_i(t)$ is the input signal
$X_i(f)$ is the FT of the input signal
$x_o(t)$ is the output signal
$X_o(f)$ is the FT of the output signal
$n_i(t)$ is the input noise signal
$N_i(f)$ is the input noise PSD
$n_o(t)$ is the out noise signal
$N_o(f)$ is the output noise PSD
The optimum filter input signal can then be represented by

$$s_i(t) = x_i(t - t_0) + n_i(t) \tag{4.1}$$

where t_0 is an unknown time delay proportional to the target range. The optimum filter output signal is

$$s_o(t) = x_o(t - t_0) + n_o(t) \tag{4.2}$$

where

$$n_o(t) = n_i(t) \otimes h(t) \tag{4.3}$$

$$x_o(t) = x_i(t) \otimes h(t) \tag{4.4}$$

The operator ($\otimes$) indicates convolution. The FT of Eq. (4.4) is

$$X_o(f) = X_i(f)H(f) \tag{4.5}$$

Consequently the signal output at time t_0 can be calculated using the inverse FT, evaluated at t_0, as

$$x_o(t_0) = \int_{-\infty}^{\infty} X_i(f)H(f)e^{j2\pi f t_0}\, df \tag{4.6}$$

Additionally, the total noise power at the output of the filter is calculated using Parseval's theorem as

$$N_o = \int_{-\infty}^{\infty} N_i(f)|H(f)|^2\, df \tag{4.7}$$

Since the output signal power at time t_0 is equal to the modulus square of Eq. (4.6), then the instantaneous SNR at time t_0 is

$$SNR(t_0) = \frac{\left| \int_{-\infty}^{\infty} X_i(f) H(f) e^{j2\pi f t_0} \, df \right|^2}{\int_{-\infty}^{\infty} N_i(f) |H(f)|^2 \, df} \tag{4.8}$$

Remember Schawrz's inequality which has the form

$$\frac{\left| \int_{-\infty}^{\infty} X_1(f) X_2(f) \, df \right|^2}{\int_{-\infty}^{\infty} |X_1(f)|^2 \, df} \le \int_{-\infty}^{\infty} |X_2(f)|^2 \, df \tag{4.9}$$

The equal sign in Eq. (4.9) applies when $X_1(f) = KX_2{}^*(f)$ for some arbitrary constant K. Apply Schawrz's inequality to Eq. (4.8) with the following assumptions

$$X_1(f) = H(f)\sqrt{N_i(f)} \tag{4.10}$$

$$X_2(f) = \frac{X_i(f) e^{j2\pi f t_0}}{\sqrt{N_i(f)}} \tag{4.11}$$

It follows that the SNR is maximized when

$$H(f) = K \frac{X_i{}^*(f) e^{-j2\pi f t_0}}{N_i(f)} \tag{4.12}$$

An alternative way of writing Eq. (4.12) is

$$X_i(f) H(f) e^{j2\pi f t_0} = K N_i(f) |X_i(f)|^2 \tag{4.13}$$

The optimum filter impulse response is computed using inverse FT integral

$$h(t) = \int_{-\infty}^{\infty} K \frac{X_i{}^*(f) e^{-j2\pi f t_0}}{N_i(f)} e^{j2\pi f t} \, df \tag{4.14}$$

A special case of great interest to radar systems is when the input noise is bandlimited white noise with PSD given by

$$N_i(f) = \eta_0/2 \quad (4.15)$$

η_0 is a constant. The transfer function for this optimum filter is then given by

$$H(f) = X_i^*(f)e^{-j2\pi f t_0} \quad (4.16)$$

where the constant K was set equal to $\eta_0/2$. It follows that

$$h(t) = \int_{-\infty}^{\infty} [X_i^*(f)e^{-j2\pi f t_0}]\ e^{j2\pi f t}\ df \quad (4.17)$$

which can be written as

$$h(t) = x_i^*(t_0 - t) \quad (4.18)$$

Observation of Eq. (4.18) indicates that the impulse response of the optimum filter is matched to the input signal, and thus, the term *matched filter* is used for this special case. Under these conditions, the maximum instantaneous SNR at the output of the matched filter is

$$SNR(t_0) = \frac{\left|\int_{-\infty}^{\infty} X_i(f)H(f)e^{j2\pi f t_0}\ df\right|^2}{\eta_0/2} \quad (4.19)$$

and using Parseval's theorem the numerator in Eq. (4.19) is equal to the input signal energy, E_x; consequently one can write the output peak instantaneous SNR as

$$SNR(t_0) = \frac{2E_x}{\eta_0} \quad (4.20)$$

Note that Eq. (4.20) is unitless since the unit for η_0 are in Watts per Hertz (or Joules). Finally, one can draw the conclusion that the peak instantaneous SNR depends only on the signal energy and input noise power, and is independent of the waveform utilized by the radar.

As indicated by Eq. (4.18) the impulse response $h(t)$ may not be causal if the value for t_0 is less than the signal duration. Thus, an additional time delay term $\tau_0 \geq T$ is added to ensure causality, where T is the signal duration. Thus, a realizable matched filter response is given by

$$h(t) = \begin{cases} x_i^*(\tau_0 + t_0 - t) & ;t > 0,\ \tau_0 \geq T \\ 0 & ;t < 0 \end{cases} \tag{4.21}$$

The transfer function for this casual filter is

$$H(f) = \int_{-\infty}^{\infty} x_i^*(\tau_0 + t_0 - t)e^{-j2\pi ft}dt = \int_{\infty}^{-\infty} x_i^*(t + \tau_0 + t_0)e^{j2\pi ft}dt \tag{4.22}$$

$$= X_i^*(f)e^{-j2\pi f(\tau_0 + t_0)}$$

Substituting the right-hand side of Eq. (4.22) into Eq. (4.6) yields

$$x_o(\tau_0) = \int_{-\infty}^{\infty} X_i(f)X_i^*(f)e^{-j2\pi f(\tau_0 + t_0)}e^{j2\pi ft_0}\ df = \int_{-\infty}^{\infty} |X_i(f)|^2 e^{-j2\pi f\tau_0}\ df \tag{4.23}$$

which has a maximum value when τ_0. This result leads to the following conclusion: The peak value of the matched filter output is obtained by sampling its output at times equal to the filter delay after the start of the input signal, and the minimum value for τ_0 is equal to the signal duration T.

Example:

Compute the maximum instantaneous SNR at the output of a linear filter whose impulse response is matched to the signal $x(t) = \exp(-t^2/2T)$.

Solution:

The signal energy is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{(-t^2)/T} dt = \sqrt{\pi T}\ \ Joules$$

It follows that the maximum instantaneous SNR is

$$SNR = \frac{\sqrt{\pi T}}{\eta_0/2}$$

where $\eta_0/2$ *is the input noise power spectrum density.*

4.1.1. The Replica

Again, consider a radar system that uses a finite duration energy signal $x(t)$, and assume that a matched filter receiver is utilized. From Eq. (4.1) the input signal can be written as,

$$s(t) = x(t - t_0) + n(t) \tag{4.24}$$

The matched filter output $s_o(t)$ can be expressed by the convolution integral between the filter's impulse response and $s(t)$:

$$s_0(t) = \int_{-\infty}^{\infty} s(u)h(t-u)du \tag{4.25}$$

Substituting Eq. (4.21) into Eq. (4.25) yields

$$s_o(t) = \int_{-\infty}^{\infty} s(u)x^*(t - \tau_0 - t_0 + u)du = \bar{R}_{sx}(t - T_0) \tag{4.26}$$

where $T_0 = \tau_0 + t_0$ and $\bar{R}_{sx}(t - T_0)$ is a cross-correlation between $s(t)$ and $x(T_0 - t)$. Therefore, the matched filter output can be computed from the cross-correlation between the radar received signal and a delayed replica of the transmitted waveform. If the input signal is the same as the transmitted signal, the output of the matched filter would be the autocorrelation function of the received (or transmitted) signal. In practice, replicas of the transmitted waveforms are normally computed and stored in memory for use by the radar signal processor when needed.

4.2. Mean and Variance of the Matched Filter Output

Since the matched filter is an LTI filter, then when its input's statistics is Gaussian, its output statistics is also Gaussian, as discussed in Chapter 3. For this purpose, consider the following two hypotheses. Hypothesis H_0 is when the input to the matched filter consists of noise only. That is,

$$H_0 \Leftrightarrow s(t) = n_i(t) \tag{4.27}$$

where $n_i(t)$ is zero mean Gaussian bandlimited white noise with PSD $\eta_0/2$. Hypothesis H_1 is when the input consists of signal plus noise. That is,

$$H_1 \Leftrightarrow s(t) = x_i(t) + n_i(t) \tag{4.28}$$

Denote the conditional means and variances for both hypotheses by $E[s_o/H_0]$, the mean value of $s_0(\tau_0)$, when the signal is absent; $E[s_o/H_1]$ is the mean value of $s_0(\tau_0)$ when the signal is present; $Var[s_o/H_0]$ is the variance of $s_0(\tau_0)$ when the signal is absent; and $Var[s_o/H_1]$ is the variance of $s_0(\tau_0)$ when the signal is present. It follows that

$$E[s_o/H_0] = 0 \tag{4.29}$$

$$E[s_o/H_1] = \int_{-\infty}^{\infty} |x_i(t)|^2 dt = E_x \tag{4.30}$$

where E_x is the signal energy. Finally,

$$Var[s_o/H_0] = Var[s_o/H_1] = E_x \eta_0 / 2 \tag{4.31}$$

4.3. General Formula for the Output of the Matched Filter

Two cases are analyzed; the first is when a stationary target is present. The second case is concerned with a moving target whose velocity is constant. Assume the range to the target is

$$R(t) = R_0 - v(t - t_0) \tag{4.32}$$

where v is the target radial velocity (i.e. the target velocity component on the radar line of sight.) The initial detection range R_0 is given by

$$t_0 = \frac{2R_0}{c} \tag{4.33}$$

where c is the speed of light and t_0 is the round trip delay it takes a certain radar pulse to travel from the radar to the target at range R_0 and back.

The general expression for the radar bandpass signal is

$$s(t) = s_I(t)\cos 2\pi f_0 t - s_Q(t)\sin 2\pi f_0 t \tag{4.34}$$

which can be written using its pre-envelope (analytic signal) as

$$s(t) = Re\{\psi(t)\} = Re\{\tilde{s}(t)e^{j2\pi f_0 t}\} \tag{4.35}$$

where $Re\{\ \}$ indicates "the real part of." Again $\tilde{s}(t)$ is the complex envelope.

4.3.1. Stationary Target Case

In this case, the received radar return is given by

$$s_r(t) = s\left(t - \frac{2R_0}{c}\right) = s(t - t_0) = Re\{\tilde{s}(t - t_0)e^{j2\pi f_0(t - t_0)}\} \quad \textbf{(4.36)}$$

It follows that the received analytic and complex envelope signals are, respectively, given by

$$\psi_r(t) = \tilde{s}(t - t_0)e^{-j2\pi f_0 t_0}e^{j2\pi f_0 t} \quad \textbf{(4.37)}$$

$$\tilde{s}_r(t) = \tilde{s}(t - t_0)e^{-j2\pi f_0 t_0} \quad \textbf{(4.38)}$$

Observation of Eq. (4.38) clearly indicates that the received complex envelope is more than just a delayed version of the transmitted complex envelope. It actually contains an additional phase shift φ_0 which represents the phase corresponding to the two-way optical length for the target range. That is,

$$\varphi_0 = -2\pi f_0 t_0 = -2\pi f_0 2\frac{R_0}{c} = -\frac{2\pi}{\lambda}2R_0 \quad \textbf{(4.39)}$$

where λ is the radar wavelength and is equal to c/f_0. Since a very small change in range can produce significant change in this phase term, this phase is often treated as a random variable with uniform probability density function over the interval $\{0, 2\pi\}$. Furthermore, the radar signal processor will first attempt to remove (correct for) this phase term through a process known as phase unwrapping.

Substituting Eq. (4.38) into Eq. (4.25) provides the output of the matched filter. It is given by

$$s_o(t) = \int_{-\infty}^{\infty} \tilde{s}_r(u)h(t - u)du \quad \textbf{(4.40)}$$

where the impulse response $h(t)$ is in Eq. (4.18). It follows that

$$s_o(t) = \int_{-\infty}^{\infty} \tilde{s}(u - t_0)e^{-j2\pi f_0 t_0}\tilde{s}^*(t - t_0 + u)du \quad \textbf{(4.41)}$$

Make the following change of variables:

$$z = u - t_0 \Rightarrow dz = du \quad \textbf{(4.42)}$$

Therefore, the output of the matched filter when a stationary target is present is computed from Eq (4.41) as

$$s_o(t) = e^{-j2\pi f_0 t_0} \int_{-\infty}^{\infty} \tilde{s}(z)\tilde{s}^*(t-z)dz = e^{-j2\pi f_0 t_0}\bar{R}_s(t) \tag{4.43}$$

where $\bar{R}_s(t)$ is the autocorrelation function for the signal $\tilde{s}(t)$.

4.3.2. Moving Target Case

In this case, the received signal only not is delayed in time by t_0 but also has a Doppler frequency shift f_d corresponding to the target velocity, where

$$f_d = 2vf_0/c = 2v/\lambda \tag{4.44}$$

The pre-envelope of the received signal can be written as

$$\psi_r(t) = \psi\left(t - \frac{2R(t)}{c}\right) = \tilde{s}\left(t - \frac{2R(t)}{c}\right)e^{j2\pi f_0\left(t - \frac{2R(t)}{c}\right)} \tag{4.45}$$

Substituting Eq. (4.32) into Eq. (4.45) yields

$$\psi_r(t) = \tilde{s}\left(t - \frac{2R_0}{c} + \frac{2vt}{c} - \frac{2vt_0}{c}\right)e^{j2\pi f_0\left(t - \frac{2R_0}{c} + \frac{2vt}{c} - \frac{2vt_0}{c}\right)} \tag{4.46}$$

Collecting terms yields

$$\psi_r(t) = \tilde{s}\left(t\left(1 + \frac{2v}{c}\right) - t_0\left(1 + \frac{2v}{c}\right)\right)e^{j2\pi f_0\left(t - \frac{2R_0}{c} + \frac{2vt}{c} - \frac{2vt_0}{c}\right)} \tag{4.47}$$

Define the scaling factor γ as

$$\gamma = 1 + \frac{2v}{c} \tag{4.48}$$

then Eq. (4.47) can be written as

$$\psi_r(t) = \tilde{s}(\gamma(t - t_0))e^{j2\pi f_0\left(t - \frac{2R_0}{c} + \frac{2vt}{c} - \frac{2vt_0}{c}\right)} \tag{4.49}$$

Since $c \gg v$, the following approximation can be used

$$\tilde{s}(\gamma(t - t_0)) \approx \tilde{s}(t - t_0) \tag{4.50}$$

It follows that Eq. (4.49) can now be rewritten as

$$\psi_r(t) = \tilde{s}(t - t_0)e^{j2\pi f_0 t}e^{-j2\pi f_0\frac{2R_0}{c}}e^{j2\pi f_0\frac{2vt}{c}}e^{-j2\pi f_0\frac{2vt_0}{c}} \tag{4.51}$$

Recognizing that $f_d = (2vf_0)/c$ and $t_0 = (2R_0)/c$, the received pre-envelope signal is

$$\psi_r(t) = \tilde{s}(t-t_0)e^{j2\pi f_0 t}e^{-j2\pi f_0 t_0}e^{j2\pi f_d t}e^{-j2\pi f_d t_0} = \tilde{s}(t-t_0)e^{j2\pi(f_0+f_d)(t-t_0)} \quad \textbf{(4.52)}$$

or

$$\psi_r(t) = \{\tilde{s}(t-t_0)e^{j2\pi f_d t}e^{-j2\pi(f_0+f_d)t_0}\}e^{j2\pi f_0 t} \quad \textbf{(4.53)}$$

Then by inspection the complex envelope of the received signal is

$$\tilde{s}_r(t) = \tilde{s}(t-t_0)e^{j2\pi f_d t}e^{-j2\pi(f_0+f_d)t_0} \quad \textbf{(4.54)}$$

Finally, it is concluded that the complex envelope of the received signal when the target is moving at a constant velocity v is a delayed (by t_0) version of the complex envelope signal of the stationary target case except that:

1. An additional phase shift term corresponding to the target's Doppler frequency is present, and
2. The phase shift term $(-2\pi f_d t_0)$ is present.

The output of the matched filter was derived in Eq. (4.25). Substituting Eq. (4.54) into Eq. (4.25) yields

$$s_o(t) = \int_{-\infty}^{\infty}\tilde{s}(u-t_0)e^{j2\pi f_d u}e^{-j2\pi(f_0+f_d)t_0}\tilde{s}^*(t-t_0+u)\ du \quad \textbf{(4.55)}$$

Applying the change of variables given in Eq. (4.42) and collecting terms provide

$$s_o(t) = e^{-j2\pi f_0 t_0}\int_{-\infty}^{\infty}\tilde{s}(z)\tilde{s}^*(t-z)e^{j2\pi f_d z}e^{j2\pi f_d t_0}e^{-j2\pi f_d t_0}\ dz \quad \textbf{(4.56)}$$

Observation of Eq. (4.56) shows that the output is a function of both t and f_d. Thus, it is more appropriate to rewrite the output of the matched filter as a two-dimensional function of both variables. That is,

$$s_o(t;f_d) = e^{-j2\pi f_0 t_0}\int_{-\infty}^{\infty}\tilde{s}(z)\tilde{s}^*(t-z)e^{j2\pi f_d z}\ dz \quad \textbf{(4.57)}$$

It is customary but not necessary to set $t_0 = 0$. Note that if the causal impulse response is used (i.e., Eq. (4.21)), the same analysis will hold true. However, in this case, the phase term is equal to $\exp(-j2\pi f_0 T_0)$, instead of $\exp(-j2\pi f_0 t_0)$ where $T_0 = \tau_0 + t_0$.

4.4. Waveform Resolution and Ambiguity

As indicated by Eq. (4.20), the radar sensitivity (in the case of white additive noise) depends only on the total energy of the received signal and is independent of the shape of the specific waveform. This leads to the following question: If the radar sensitivity is independent of the waveform, what is the best choice for the transmitted waveform? The answer depends on many factors; however, the most important consideration lies in the waveform's range and Doppler resolution characteristics, which can be determined from the output of the matched fitter.

As discussed in Chapter 1, range resolution implies separation between distinct targets in range. Alternatively, Doppler resolution implies separation between distinct targets in frequency. Thus, ambiguity and accuracy of this separation are closely associated terms.

4.4.1. Range Resolution

Consider radar returns from two stationary targets (zero Doppler) separated in range by distance ΔR. What is the smallest value of ΔR so that the returned signal is interpreted by the radar as two distinct targets? In order to answer this question, assume that the radar transmitted bandpass pulse is denoted by $x(t)$,

$$x(t) = r(t)\cos(2\pi f_0 t + \phi(t)) \tag{4.58}$$

where f_0 is the carrier frequency, $r(t)$ is the amplitude modulation, and $\phi(t)$ is the phase modulation. The signal $x(t)$ can then be expressed as the real part of the pre-envelope signal $\psi(t)$, where

$$\psi(t) = r(t)e^{j(2\pi f_0 t - \phi(t))} = \tilde{x}(t)e^{2\pi f_0 t} \tag{4.59}$$

and the complex envelope is

$$\tilde{x}(t) = r(t)e^{-j\phi(t)} \tag{4.60}$$

It follows that

$$x(t) = Re\{\psi(t)\} \tag{4.61}$$

The returns from two close targets are, respectively, given by

$$x_1(t) = \psi(t - \tau_0) \tag{4.62}$$

$$x_2(t) = \psi(t - \tau_0 - \tau) \tag{4.63}$$

where τ is the difference in delay between the two target returns. One can assume that the reference time is τ_0, and thus without any loss of generality, one may set $\tau_0 = 0$. It follows that the two targets are distinguishable by how large or small the delay τ can be.

In order to measure the difference in range between the two targets, consider the integral square error between $\psi(t)$ and $\psi(t-\tau)$. Denoting this error as ε_R^2, it follows that

$$\varepsilon_R^2 = \int_{-\infty}^{\infty} |\psi(t) - \psi(t-\tau)|^2 \; dt \tag{4.64}$$

which can be written as

$$\varepsilon_R^2 = \int_{-\infty}^{\infty} |\psi(t)|^2 \; dt + \int_{-\infty}^{\infty} |\psi(t-\tau)|^2 \; dt - \int_{-\infty}^{\infty} \{(\psi(t)\psi^*(t-\tau) + \psi^*(t)\psi(t-\tau)) \; dt\} \tag{4.65}$$

Using Eq. (4.59) into Eq. (4.65) yields

$$\varepsilon_R^2 = 2\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 \; dt - 2Re\left\{\int_{-\infty}^{\infty} \psi^*(t)\psi(t-\tau) \; dt\right\} = 2\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 \; dt - 2Re\left\{e^{-j\omega_0\tau}\int_{-\infty}^{\infty} \tilde{x}^*(t)\tilde{x}(t-\tau) \; dt\right\} \tag{4.66}$$

This squared error is minimum when the second portion of Eq. (4.66) is positive and maximum. Note that the first term in the right-hand side of Eq. (4.66) represents the total signal energy, and is assumed to be constant. The second term is a varying function of τ with its fluctuation tied to the carrier frequency. The integral inside the right most side of this equation is defined as the range ambiguity function,

$$\chi_R(\tau) = \int_{-\infty}^{\infty} \tilde{x}^*(t)\tilde{x}(t-\tau) \; dt \tag{4.67}$$

This range ambiguity function is equivalent to the integral given in Eq. (4.43) with $t_0 = 0$. Comparison between Eq. (4.67) and Eq. (4.43) indicates that the output of the matched filter and the range ambiguity function have the same envelope (in this case the Doppler shift f_d is set to zero). This indicates that the matched filter, in addition to providing the maximum instantaneous SNR at its output, also preserves the signal range resolution properties. The value of $\chi_R(\tau)$ that minimizes the squared error in Eq. (4.66) occurs when $\tau = 0$.

Target resolvability in range is measured by the squared magnitude $|\chi_R(\tau)|^2$. It follows that if $|\chi_R(\tau)| = \chi_R(0)$ for some nonzero value of τ, then the two targets are indistinguishable. Alternatively, if $|\chi_R(\tau)| \neq \chi_R(0)$ for some nonzero value of τ, then the two targets may be distinguishable (resolvable). As a consequence, the most desirable shape for $\chi_R(\tau)$ is a very sharp peak (thumb tack shape) centered at $\tau = 0$ and falling very quickly away from the peak. The minimum range resolution corresponding to a time duration τ_e or effective bandwidth B_e is

$$\Delta R = \frac{c\tau_e}{2} = \frac{c}{2B_e} \tag{4.68}$$

The effective time duration and the effective bandwidth for any waveform were defined in Chapter 2 and are repeated here as Eq. (4.69) and Eq. (4.70), respectively

$$\tau_e = \left[\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 dt\right]^2 \Big/ \int_{-\infty}^{\infty} |\tilde{x}(t)|^4 dt \tag{4.69}$$

$$B_e = \left[\int_{-\infty}^{\infty} |\tilde{X}(f)|^2 \ df\right]^2 \Big/ \left(\int_{-\infty}^{\infty} |\tilde{X}(f)|^4 \ df\right) \tag{4.70}$$

4.4.2. Doppler Resolution

The Doppler shift corresponding to the target radial velocity is

$$f_d = \frac{2v}{\lambda} = \frac{2vf_0}{c} \tag{4.71}$$

where v is the target radial velocity, λ is the wavelength, f_0 is the frequency, and c is the speed of light.

The FT of the pre-envelope is

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi ft} \; dt \tag{4.72}$$

Due to the Doppler shift associated with the target, the received signal spectrum will be shifted by f_d. In other words, the received spectrum can be represented by $\Psi(f - f_d)$. In order to distinguish between the two targets located at the same range but having different velocities, one may use the integral square error. More precisely,

$$\varepsilon_f^2 = \int_{-\infty}^{\infty} |\Psi(f) - \Psi(f - f_d)|^2 \; df \tag{4.73}$$

Using similar analysis as that which led to Eq. (4.66), one should maximize

$$Re\left\{ \int_{-\infty}^{\infty} \Psi^*(f)\Psi(f - f_d) \; df \right\} \tag{4.74}$$

Taking the FT of the pre-envelope (analytic signal) defined in Eq. (4.59) yields

$$\Psi(f) = \tilde{X}(2\pi f - 2\pi f_0) \tag{4.75}$$

Thus,

$$\int_{-\infty}^{\infty} \tilde{X}^*(2\pi f)\tilde{X}(2\pi f - 2\pi f_d) \; df = \tag{4.76}$$

$$\int_{-\infty}^{\infty} \tilde{X}^*(2\pi f - 2\pi f_0)\tilde{X}(2\pi f - 2\pi f_0 - 2\pi f_d) \; df$$

The complex frequency correlation function is then defined as

$$\chi_f(f_d) = \int_{-\infty}^{\infty} \tilde{X}^*(2\pi f)\tilde{X}(2\pi f - 2\pi f_d) \; df = \int_{-\infty}^{\infty} |\tilde{x}(t)|^2 \; {}^{j2\pi f_d t} \; dt \tag{4.77}$$

The velocity resolution (Doppler resolution) is by definition

$$\Delta v = (c\Delta f_d)/(2f_0) \tag{4.78}$$

where Δf_d is the minimum resolvable Doppler difference between the Doppler frequencies corresponding to two moving targets, i.e., $\Delta f_d = f_{d1} - f_{d2}$, where

f_{d1} and f_{d2} are the two individual Doppler frequencies for targets 1 and 2, respectively. The Doppler resolution Δf_d is equal to the inverse of the total effective duration of the waveform. Thus,

$$\Delta f_d = \left(\int_{-\infty}^{\infty} |\chi_f(f_d)|^2 df_d \right) / (\chi_f^2(0)) = \left(\int_{-\infty}^{\infty} |\tilde{x}(t)|^4 dt \right) / \left[\int_{-\infty}^{\infty} |\tilde{x}(t)|^2 dt \right]^2 = \frac{1}{\tau_e} \quad \textbf{(4.79)}$$

4.4.3. Combined Range and Doppler Resolution

In this general case, one needs to use a two-dimensional function in the pair of variables (τ, f_d). For this purpose, assume that the pre-envelope of the transmitted waveform is

$$\psi(t) = \tilde{x}(t) e^{j2\pi f_0 t} \quad \textbf{(4.80)}$$

Then the delayed and Doppler-shifted signal is (see Eq. (4.53))

$$\psi(t-\tau) = \tilde{x}(t-\tau) e^{j2\pi(f_0 - f_d)(t-\tau)} \quad \textbf{(4.81)}$$

Computing the integral square error between Eq. (4.80) and Eq. (4.81) yields

$$\varepsilon^2 = \int_{-\infty}^{\infty} |\psi(t) - \psi(t-\tau)|^2 dt \quad \textbf{(4.82a)}$$

$$\varepsilon^2 = 2 \int_{-\infty}^{\infty} |\psi(t)|^2 dt - 2Re \left\{ \int_{-\infty}^{\infty} \psi^*(t) - \psi(t-\tau) dt \right\} \quad \textbf{(4.82b)}$$

which can be written as

$$\varepsilon^2 = 2 \int_{-\infty}^{\infty} |\tilde{x}(t)|^2 \; dt - 2Re \left\{ e^{j2\pi(f_0 - f_d)\tau} \int_{-\infty}^{\infty} \tilde{x}(t) \tilde{x}^*(t-\tau) e^{j2\pi f_d t} dt \right\} \quad \textbf{(4.83)}$$

Again, in order to maximize this squared error for $\tau \neq 0$, one must minimize the last term of Eq. (4.83). Define the combined range and Doppler correlation function as

$$\chi(\tau, f_d) = \int_{-\infty}^{\infty} \tilde{x}(t) \tilde{x}^*(t-\tau) e^{j2\pi f_d t} dt \quad \textbf{(4.84)}$$

In order to achieve the most range and Doppler resolution, the modulus square of this function must be minimized at $\tau \neq 0$ and $f_d \neq 0$. Note that the output of the matched filter, except for a phase term, is identical to that given in Eq. (4.84). This means that the output of the filter exhibits maximum instantaneous SNR as well as the most achievable range and Doppler resolutions. The modulus square of Eq. (4.84) is often referred to as the ambiguity function:

$$|\chi(\tau, f_d)|^2 = \left| \int_{-\infty}^{\infty} \tilde{x}(t)\tilde{x}^*(t-\tau)e^{j2\pi f_d t}\, dt \right|^2 \tag{4.85}$$

The ambiguity function is often used by radar designers and analysts to determine the *goodness* of a given radar waveform, where this *goodness* is measured by its range and Doppler resolutions. Remember that since the matched filter is used, maximum SNR is guaranteed.

4.5. Range and Doppler Uncertainty

The formula derived in Eq. (4.84) represents the output of the matched filter when the signal at its input comprises target returns only and has no noise components, an assumption that cannot be true in practical situations. In general, the input at the matched filter contains both target and noise returns. The noise signal is assumed to be an additive random process that is uncorrelated with the target and has bandlimited white spectrum. Referring to Eq. (4.84), a peak at the output of the matched filter at (τ_1, f_{d1}) represents a target whose delay (range) corresponds to τ_1 and Doppler frequency equal to f_{d1}. Therefore, measuring targets' exact range and Doppler frequency is determined from measuring peak locations occurring in the two-dimensional space (τ, f_d). This last statement, however, is correct only if noise is not present at the input of the matched filter. When noise is present and because noise is random, it will generate ambiguity (uncertainty) about the exact location of the ambiguity function peaks in the (τ, f_d) space.

4.5.1. Range Uncertainty

Consider the case when the return signal complex envelope is (assuming stationary target)

$$\tilde{s}_r(t) = \tilde{x}_r(t) + \tilde{n}(t) \tag{4.86}$$

where $\tilde{x}_r(t)$ is the target return signal complex envelope and $\tilde{n}(t)$ is the noise signal complex envelope. The integral squared error between the total received signal (target plus noise) and the shifted (delayed) transmitted waveform is

$$\varepsilon^2 = \int_0^{T_{max}} \left|\tilde{x}(t-\tau) - \tilde{s}_r(t)\right|^2 \ dt \tag{4.87}$$

where T_{max} corresponds to maximum range under consideration. Expanding this squared error yields

$$\varepsilon^2 = 2\int_0^{T_{max}} |\tilde{x}(t)|^2 \ dt + 2\int_0^{T_{max}} |\tilde{n}(t)|^2 \ dt - 2Re\left\{\int_0^{T_{max}} \tilde{x}^*(t-\tau)\tilde{s}_r(t)dt\right\} \tag{4.88}$$

which can be written as

$$\varepsilon^2 = E_x + E_n - 2Re\left\{\int_0^{T_{max}} \tilde{x}^*(t-\tau)\tilde{x}_r(t)dt + \int_0^{T_{max}} \tilde{x}^*(t-\tau)\tilde{n}(t)dt\right\} \tag{4.89}$$

This expression is minimum at some value τ that makes the integral term inside Eq. (4.88) maximum and positive. More precisely, the following correlation functions must be maximized

$$R_{x_r x}(\tau) = \int_0^{T_{max}} \tilde{x}^*(t-\tau)\tilde{x}_r(t)dt \tag{4.90}$$

$$R_{nx}(\tau) = \int_0^{T_{max}} \tilde{x}^*(t-\tau)\tilde{n}(t)dt \tag{4.91}$$

Therefore, Eq. (4.89) can be written as

$$\varepsilon^2 = E - 2Re\{R_{x_r x}(\tau) + R_{nx}(\tau)\} \tag{4.92}$$

Expanding the quantity $\{R_{x_r x}(\tau)\}$ using Taylor series expansion about the point $\tau = t_0$, where $t_0 = 2R/c$, and R is the exact target range leads to

$$R_{x_r x}(\tau) = R_{x_r x}(t_0) + R'_{x_r x}(t_0)(\tau - t_0) + \frac{R''_{x_r x}(t_0)(\tau - t_0)^2}{2!} + \ldots \tag{4.93}$$

where R' and R'', respectively, indicate the first and second derivatives with respect to delay. Remember that since the real part of the correlation function is an even function, all its odd number derivatives are equal to zero. Now, approximate Eq. (4.93) by using the first three terms (terms 1 and 3 are, of course, equal to zero) to get

$$Re\{R_{x_rx}(\tau)\} \approx R_{x_rx}(t_0) + \frac{R''_{x_rx}(t_0)(\tau - t_0)^2}{2} \tag{4.94}$$

There is some value τ_1 close to the exact target range, t_0, that will minimize the expression in Eq. (4.92). In order to find this minimum value, differentiate the quantity $Re\{R_{x_rx}(\tau) + R_{nx}(\tau)\}$ with respect to τ and set the result equal to zero to find τ_1. More specifically,

$$Re\left\{\frac{d}{d\tau}R_{x_rx}(\tau) + \frac{d}{d\tau}R_{nx}(\tau)\right\} = Re\{R'_{x_rx}(\tau) + R'_{nx}(\tau)\} = 0 \tag{4.95}$$

The derivative of the $Re\{R_{x_rx}(\tau)\}$ can be found from Eq. (4.94) as

$$Re\left\{\frac{d}{d\tau}R_{x_rx}(\tau)\right\} = \frac{d}{d\tau}\left(R_{x_rx}(t_0) + \frac{R''_{x_rx}(t_0)(\tau - t_0)^2}{2!}\right) = R''_{x_rx}(t_0)(\tau - t_0) \tag{4.96}$$

Substituting the result of Eq. (4.96) into Eq. (4.95) and collecting terms yield

$$(\tau_1 - t_0) = -\frac{Re\{R'_{nx}(\tau_1)\}}{R''_{x_rx}(t_0)} \tag{4.97}$$

The value $(\tau_1 - t_0)$ represent the amount of target range error measurement. It is more meaningful, since noise is random, to compute this error in terms of the standard deviation of its rms value. Hence, the standard deviation for range measurement error is

$$\sigma_\tau = (\tau_1 - t_0)_{rms} = -\frac{Re\{R'_{nx}(\tau_1)\}_{rms}}{R''_{x_rx}(t_0)} \tag{4.98}$$

By using the differentiation property of the Fourier transform and Parseval's theorem the denominator of Eq. (4.89) can be determined by

$$R''_{x_rx}(t_0) = (2\pi)^2 \int_{-\infty}^{\infty} f^2 \; |X(f)|^2 df \tag{4.99}$$

Next, from relations developed in Chapter 2, one can write the FT of $R_{nx}(\tau)$ as

$$FT\{R_{nx}(\tau)\} = X^*(f)\frac{\eta_0}{2} \tag{4.100}$$

where $\eta_0/2$ is the noise power spectrum density value (white noise). From the Fourier transform properties, the FT of the derivative of $R_{nx}(\tau)$ is

$$FT\{R'_{nx}(\tau)\} = (j2\pi f)\left(X^*(f)\frac{\eta_0}{2}\right) = (j2\pi f)S_{nx}(f) \tag{4.101}$$

The rms value for $R'_{nx}(\tau)$ is by definition

$$\{R'_{nx}(\tau)\}_{rms} = \sqrt{\lim_{T_{max}} \frac{1}{T_{max}} \int_0^{T_{max}} R'_{nx}(\tau)\ d\tau} \tag{4.102}$$

which can be rewritten using Parseval's theorem as

$$\{R'_{nx}(\tau)\}_{rms} = \sqrt{\int_0^{T_{max}} |FT\{R'_{nx}(\tau)\}|^2\ df} \tag{4.103}$$

substituting Eq. (4.101) into Eq. (4.103) yields

$$\{R'_{nx}(\tau)\}_{rms} = \sqrt{\frac{\eta_0}{2}(2\pi)^2 \int_0^{T_{max}} f^2\ |X(f)|^2\ df} \tag{4.104}$$

Finally, the standard deviation for range measurement error can be written as

$$\sigma_\tau = \frac{\sqrt{\eta_0/2}}{\sqrt{(2\pi)^2 \int_{-\infty}^{\infty} f^2\ |X(f)|^2 df}} \tag{4.105}$$

Define the bandwidth rms value, B^2_{rms}, as

$$B^2_{rms} = \frac{(2\pi)^2 \int_{-\infty}^{\infty} f^2\ |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} \tag{4.106}$$

It follows that Eq. (4.105) can now be written as

$$\sigma_\tau = \frac{\sqrt{\eta_0/2}}{B_{rms}\sqrt{\int_{-\infty}^{\infty}|X(f)|^2 df}} = \frac{\sqrt{\eta_0/2}}{B_{rms}\sqrt{E_x}} = \frac{1}{B_{rms}\sqrt{2E_x/\eta_0}} \tag{4.107}$$

which leads to the conclusion that the uncertainty in range measurement is inversely proportional to the rms bandwidth and the square root of the ratio of signal energy to the noise power density (square root of the SNR).

4.5.2. Doppler (Velocity) Uncertainty

For this purpose, assume that the target range is completely known. In the next section the case where both target range and target Doppler are not known will be analyzed. Denote the signal transmitted by the radar as $x(t)$ and the received signal (target plus noise) as $x_r(t)$. The integral square difference between the two returns can be written as

$$\varepsilon^2 = \int_0^{f_{max}} |X(f-f_c) - X_r(f)|^2 \, df \tag{4.108}$$

where $X(f)$ is the FT of $x(t)$, $X_r(f)$ is the FT of $x_r(t)$, and f_{max} is the maximum anticipated target Doppler. Again expand Eq. (4.108) to get

$$\varepsilon^2 = \int_0^{f_{max}} |X(f)|^2 \, df + \int_0^{f_{max}} |X_r(f)|^2 \, df - 2Re\left\{\int_0^{f_{max}} |X^*(f-f_c)X_r(f)|^2 \, df\right\} \tag{4.109}$$

Minimizing the error squared in Eq. (4.109) requires maximizing the value

$$Re\left\{\int_0^{f_{max}} |X^*(f-f_c)X_r(f)|^2 \, df\right\}$$

Conducting similar analysis as that performed in the previous section, the duration rms, τ_{rms}^2, value can be defined as

$$\tau_{rms}^2 = \left((2\pi)^2 \int_{-\infty}^{\infty} t^2 \, |x(t)|^2 dt\right) \Big/ \left(\int_{-\infty}^{\infty} |x(t)|^2 dt\right) \tag{4.110}$$

The standard deviation in the Doppler measurement can be derived as

$$\sigma_{f_d} = \frac{1}{\tau_{rms}\sqrt{2E_x/\eta_0}} \tag{4.111}$$

Comparison of Eq. (4.111) and Eq. (4.107) indicates that the error in estimating Doppler is inversely proportional to the signal duration, while the error in estimating range is inversely proportional to the signal bandwidth. Therefore, and as expected, larger bandwidths minimize the range measurement errors and longer integration periods minimize the Doppler measurement errors.

4.5.3. Range-Doppler Coupling

In the previous two sections, range estimate error and Doppler estimate error were derived by assuming that they are uncoupled estimates. In other words, range error was derived assuming stationary target, while Doppler error was derived assuming completely known target range. In this section a more general formula for the combined range and Doppler errors is derived.

The analytic signal for this case was derived in Section 4.3 and was given in Eq. (4.52) which is repeated here as Eq. (4.112) for easy reference:

$$\psi_r(t) = \tilde{s}(t-t_0)e^{j2\pi f_0 t}e^{-j2\pi f_0 t_0}e^{j2\pi f_d t}e^{-j2\pi f_d t_0} = \tilde{s}(t-t_0)e^{j2\pi(f_0+f_d)(t-t_0)} \tag{4.112}$$

One can assume with any loss of generality that $t_0 = 0$, thus, Eq. (4.112) can be expressed as

$$\psi_r(t) = \tilde{s}(t)e^{j2\pi(f_0+f_d)t} = r(t)e^{j\varphi(t)}e^{j2\pi(f_0+f_d)t} \tag{4.113}$$

where the complex envelope signal, $\tilde{s}(t)$, can be expressed as

$$\tilde{s}(t) = r(t)e^{j\varphi(t)} \tag{4.114}$$

Range Error Estimate

From the analysis performed in Section 4.5.1, the estimate for the range error is determined by maximizing the function

$$Re\{R_{ss}(\tau,f_d) + R_{ns}(\tau)\} \tag{4.115}$$

It follows that for some fixed value f_{d1} there is a value τ_1 close to $t_0 = 0$ that will maximize Eq. (4.115); that is,

$$Re\{R'_{ss}(\tau_1,f_{d1}) + R'_{ns}(\tau_1)\} = 0 \tag{4.116}$$

Again the Taylor series expansion of R_{ss} about $\tau = 0$ is

$$R_{ss}(\tau, f_d) = Re\left\{R_{ss}(0, f_{d1}) + R'_{ss}(0, f_{d1})(\tau) + \frac{R''_{ss}(0, f_{d1})\tau^2}{2!} + \ldots\right\} \tag{4.117}$$

Thus,

$$Re\left\{\frac{d}{d\tau}R_{ss}(\tau, f_d)\right\} \approx Re\{R'_{ss}(0, f_{d1}) + R''_{ss}(0, f_{d1})\tau\} \tag{4.118}$$

Substituting Eq. (4.118) into Eq. (4.116) and solving for τ_1 yields

$$\tau_1 = -\frac{Re\{R'_{ns}(\tau_1) + R'_{ss}(0, f_{d1})\}}{Re\{R''_{ss}(0, f_{d1})\}} \tag{4.119}$$

The value of $R''_{ss}(0, f_{d1})$ is not much different from $R''_{ss}(0, 0)$; thus,

$$\tau_1 \approx -\frac{Re\{R'_{ns}(\tau_1) + R'_{ss}(0, f_{d1})\}}{R''_{ss}(0, 0)} \tag{4.120}$$

To evaluate the term $R'_{ss}(0, f_{d1})$, start with the definition of $R_{ss}(\tau, f_d)$,

$$R_{ss}(\tau, f_d) = \int_{-\infty}^{\infty} r(t-\tau)e^{-j\varphi(t-\tau)}r(t)e^{j(\varphi(t) + 2\pi f_d t)}dt \tag{4.121}$$

Compute the derivative of Eq. (4.121) with respect to τ

$$R'_{ss}(\tau, f_d) = -\int_{-\infty}^{\infty}\{r'(t-\tau)r(t) - j\varphi'(t-\tau)r(t-\tau)r(t)\} \times e^{j[\varphi(t) - \varphi(t-\tau) + 2\pi f_d t]}\,dt \tag{4.122}$$

Evaluating Eq. (4.122) at $\tau = 0$ and $f_d = f_{d1}$ gives

$$R'_{ss}(0, f_{d1}) = -\int_{-\infty}^{\infty}\{r'(t)r(t) - j\varphi'(t)r^2(t)\} \times e^{j[2\pi f_{d1} t]}\,dt \tag{4.123}$$

The complex exponential term in Eq. (4.123) can be approximated using small angle approximation as

$$e^{j[2\pi f_{d1} t]} = \cos(2\pi f_{d1} t) + j\sin(2\pi f_{d1} t) \approx 1 + 2\pi f_{d1} t \tag{4.124}$$

Next substitute Eq. (4.124) into Eq. (4.123), collect terms, and compute its real part to get

$$Re\{R'_{ss}(0,f_{d1})\} = -\int_{-\infty}^{\infty} r'(t)r(t)dt - 2\pi f_{d1}\int_{-\infty}^{\infty} t\varphi'(t)r^2(t)dt \qquad \textbf{(4.125)}$$

The first integral is evaluated (using FT properties and Parseval's theorem) as

$$\int_{-\infty}^{\infty} r'(t)r(t)dt = (j2\pi)\int_{-\infty}^{\infty} f_d|R(f)|^2 df \qquad \textbf{(4.126)}$$

Remember that since the envelope function $r(t)$ is a real lowpass signal, its Fourier transform is an even function; thus, Eq. (4.126) is equal to zero. Using this result, Eq. (4.125) becomes

$$Re\{R'_{ss}(0,f_{d1})\} = -2\pi f_{d1}\int_{-\infty}^{\infty} t\varphi'(t)r^2(t)dt \qquad \textbf{(4.127)}$$

Substitute Eq. (4.127) into Eq. (4.120) to get

$$\tau_1 = -\frac{Re\{R'_{ns}(\tau_1)\} - 2\pi f_{d1}\int_{-\infty}^{\infty} t\varphi'(t)r^2(t)dt}{R''_{ss}(0, 0)} \qquad \textbf{(4.128)}$$

Equation (4.128) provides a measure for the degree of coupling between range and Doppler estimates. Clearly, if $\varphi(t) = 0 \Rightarrow \varphi'(t) = 0$, then there is zero coupling between the two estimates. Define the range-Doppler coupling constant as

$$\rho_{\tau RDC} = \left(2\pi\int_{-\infty}^{\infty} t\varphi'(t)|\tilde{s}(t)|^2 dt\right) \Big/ \left(\int_{-\infty}^{\infty} |\tilde{s}(t)|^2 dt\right) \qquad \textbf{(4.129)}$$

Doppler Error Estimate

Applying similar analysis as that performed in the preceding section to the spectral cross correlation function yields an expression for the range-Doppler coupling term. It is given by

$$\rho_{f_d RDC} = \frac{2\pi \int_{-\infty}^{\infty} f \; \Phi'(f) |\tilde{S}(f)|^2 df}{\int_{-\infty}^{\infty} |\tilde{S}(f)|^2 df} \tag{4.130}$$

where $\Phi(f)$ is the FT of $\varphi(t)$.

It can be shown that Eq. (4.129) and Eq. (4.130) are equal (see Problem 4.15). Given this result, the subscripts τ and f_d in Eq. (4.129) and Eq. (4.130) are dropped and the range-Doppler term is simply referred to as ρ_{RDC}.

4.5.4. Range-Doppler Coupling in LFM Signals

Referring to Eq. (4.113) and Eq. (4.114), the phase for an LFM signal can be expressed as

$$\varphi(t) = \mu' t^2 \tag{4.131}$$

where $\mu' = (\pi B)/\tau_0$, B is the LFM bandwidth, and τ_0 is the pulsewidth. Substituting Eq. (4.131) into Eq. (4.129) yields

$$\rho_{RDC} = \frac{4\pi\mu' \int_{-\infty}^{\infty} t^2 |\tilde{s}(t)|^2 dt}{\int_{-\infty}^{\infty} |\tilde{s}(t)|^2 dt} = \frac{\mu'}{\pi}\tau_e^2 \tag{4.132}$$

where τ_e is the effective duration. Thus,

$$\sigma_\tau^2 = \frac{(\eta_0/2)}{B_e^2 2E_x} + \frac{f_{d1}^2 \rho_{RDC}^2}{B_e^4} \tag{4.133}$$

Similarly,

$$\sigma_{f_d}^2 = \frac{(\eta_0/2)}{\tau_e^2 2E_x} + \frac{t_1^2 \rho_{RDC}^2}{\tau_e^4} \tag{4.134}$$

where f_{d1} and t_1 are constants. Since estimates of range or Doppler when noise is present cannot be 100% exact, it is better to replace these constants with their equivalent mean-squared errors. That is, let

$$f_{d1}^2 = \sigma_{fd}^2 \qquad , \quad t_1^2 = \sigma_\tau^2 \tag{4.135}$$

where σ_τ is as in Eq. (4.133) and σ_{fd} is in Eq. (4.134). Thus, Eq. (4.133) can be written as

$$\sigma_{\tau_{RDC}}^2 = \frac{(\eta_0/2)}{B_e^2 2E_x} + \frac{\rho_{RDC}^2}{B_e^4}\left(\frac{(\eta_0/2)}{\tau_e^2 2E_x} + \frac{\rho_{RDC}^2 \sigma_\tau^2}{\tau_e^4}\right) \tag{4.136}$$

which can be algebraically manipulated to get

$$\sigma_{\tau_{RDC}}^2 = \frac{(\eta_0/2)}{B_e^2 2E_x} \frac{1}{(1-(\rho_{RDC}^2/B_e^2\tau_e^2))} \tag{4.137}$$

Using similar analysis,

$$\sigma_{f_{dRDC}}^2 = \frac{(\eta_0/2)}{\tau_e^2 2E_x} \frac{1}{(1-(\rho_{RDC}^2/B_e^2\tau_e^2))} \tag{4.138}$$

These results lead to the conclusion that one can estimate target range and Doppler simultaneously only when the product of the rms bandwidth and rms duration is very large (i.e., very large time bandwidth products). This is the reason radars using LFM waveforms cannot estimate target Doppler accurately unless very large time bandwidth products are utilized. Often, the LFM waveforms are referred to as “Doppler insensitive” waveforms.

4.6. Target Parameter Estimation

Target parameters of interest to radar applications include, but are not limited to, target range (delay), amplitude, phase, Doppler, and angular location (azimuth and elevation). Target information (parameters) is typically embedded in the return signals amplitude and phase. Different classes waveforms are used by the radar signal and data processors to extract different target parameters more efficiently than others. Since radar echoes typically comprise signal plus additive noise, most if not all the target information is governed by the statistics of the input noise, whose statistical parameters most likely are not known but can be estimated. Thus, statistical estimates of the target parameters (amplitude, phase, delay, Doppler, etc.) are utilized instead of the actual corresponding measurements. The general form of the radar signal can be expressed in the following form

$$x(t) = Ar(t-t_0)\cos[2\pi(f_0+f_d)(t-t_0) + \phi(t-t_0) + \phi_0] \tag{4.139}$$

where A is the signal amplitude, $r(t)$ is the envelope lowpass signal, ϕ_0 is some constant phase, f_0 is the carrier frequency, t_0 and f_d are the target delay and Doppler, respectively. The analysis in this section closely follows Melsa and Cohen[1].

4.6.1. What Is an Estimator?

In the case of radar systems it always safe to assume, due to the central limit theorem, that the input noise is always Gaussian with mainly unknown parameters. Furthermore, one can assume that this noise is bandlimited white noise. Consequently, the primary question that needs to be answered is as follows: Given that the probability density function of the observation is known (Gaussian in this case) and given a finite number of independent measurements, can one determine an estimate of a given parameter (such as range, Doppler, amplitude, or phase)?

Let $f_X(x;\theta)$ be the *pdf* of a random variable X with an unknown parameter θ. Define the values $\{x_1, x_2, \ldots, x_N\}$ as N observed independent values of the variable X. Define the function or estimator $\hat{\theta}(x_1, x_2, \ldots, x_N)$ as an estimate of the unknown parameter θ. The bias of estimation is defined as

$$E[\hat{\theta} - \theta] = b \tag{4.140}$$

where $E[\ \ ]$ represents the "expected value of." The estimator $\hat{\theta}$ is referred to as an unbiased estimator if and only if

$$E[\hat{\theta}] = \theta \tag{4.141}$$

One of the most popular and common measures of the quality or effectiveness of an estimator is the Mean Square Deviation (MSD) referred to symbolically as $\Delta^2(\hat{\theta})$. For an unbiased estimator

$$\Delta^2(\hat{\theta}) = \sigma^2_{\hat{\theta}} \tag{4.142}$$

where $\sigma^2_{\hat{\theta}}$ is the estimator variance. It can be shown that the Cramer-Rao bound for this MSD is given by

$$\sigma^2(\hat{\theta}) \ge \sigma^2_{min}(\theta) = \frac{1}{N\int\limits_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\{f_X(x;\theta)\}\right)^2 f_X(x;\theta)\ dx} \tag{4.143}$$

The efficiency of this unbiased estimator is defined by

1. Melsa, J. L. Cohen, D. L., *Decision and Estimation Theory*, McGraw-Hill, New York, 1978.

$$\varepsilon(\hat{\theta}) = \frac{\sigma^2_{min}(\theta)}{\sigma^2(\hat{\theta})} \tag{4.144}$$

when $\varepsilon(\hat{\theta}) = 1$ the unbiased estimator is called an efficient estimate.

Consider an essentially timelimited signal $x(t)$ with effective duration τ_e and assume a bandlimited white noise with PSD $\eta_0/2$. In this case, Eq. (4.144) is equivalent to

$$\sigma^2(\hat{\theta}_i) \geq 1 \Big/ \left(\frac{2}{\eta_0} \int_0^{NT_r} \left(\frac{\partial}{\partial \theta_i} x(t) \right)^2 dt \right) \tag{4.145}$$

where $\hat{\theta}_i$ is the estimate for the i^{th} parameter of interest and T_r is the pulse repetition interval for the pulsed sequence. In the next two sections, estimates of the target amplitude and phase are derived. It must be noted that since these estimates represent independent random variables, they are referred to as uncoupled estimates; that is, the computation of one estimate does not depend on apriori knowledge of the other estimates.

4.6.2. Amplitude Estimation

The signal amplitude A in Eq. (4.139) is the parameter of interest, in this case. Taking the partial derivative of Eq. (4.139) with respect to A and squaring the result yields

$$\left(\frac{\partial}{\partial t_0} x(t) \right)^2 = (r(t-t_0)\cos[2\pi(f_0 + f_d)(t-t_0) + \phi(t-t_0) + \phi_0])^2 \tag{4.146}$$

Thus,

$$\int_0^{NT_r} \left(\frac{\partial}{\partial A} x(t) \right)^2 dt = \int_0^{NT_r} (x(t))^2 dt = NE_x \tag{4.147}$$

where E_x is the signal energy (from Parseval's theorem). Substituting Eq. (4.147) into Eq. (4.145) and collecting terms yield the variance for the amplitude estimate as

$$\sigma_A^2 \geq \frac{1}{\frac{2}{\eta_0} NE_x} = \frac{1}{N\ SNR} \tag{4.148}$$

In this case Eq. (4.20) used in Eq. (4.148) and SNR is the signal to noise ratio of the signal at the output of the matched filter. This clearly indicates that the signal amplitude estimate is improved as the SNR is increased.

4.6.3. Phase Estimation

In this case, it is desired to compute the best estimate for the signal phase ϕ_0. Again taking the partial derivative of the signal in Eq. (4.139) with respect to ϕ_0 and squaring the result yield

$$\left(\frac{\partial}{\partial \phi_0} x(t)\right)^2 = (-r(t-t_0)\sin[2\pi(f_0+f_d)(t-t_0)+\phi(t-t_0)+\phi_0])^2 \quad \textbf{(4.149)}$$

It follows that

$$\int_0^{NT_r} \left(\frac{\partial}{\partial \phi_0} x(t)\right)^2 \, dt = \int_0^{NT_r} (x(t))^2 \, dt = NE_x \quad \textbf{(4.150)}$$

Thus, the variance of the phase estimate is

$$\sigma_{\phi_0}^2 \ge \frac{1}{\frac{2}{\eta_0} NE_x} = \frac{1}{N \; SNR} \quad \textbf{(4.151)}$$

Problems

4.1. Show that the SNR at the output of the matched filter can be written as

$$SNR = \frac{2}{\alpha\pi}(S_i(\alpha))^2$$

where $\alpha = (\pi BT)/2$, B is the bandwidth, T is the pulsewidth. Assume that the radar is using unmodulated rectangular pulse of width T and that there is a target detected at range R. The value S_i is the signal power at the input of the matched filter.

4.2. Compute the frequency response for the filter matched to the signal
(a) $x(t) = \exp\left(\frac{-t^2}{2T}\right)$;
(b) $x(t) = u(t)\exp(-\alpha t)$ where α is a positive constant.

4.3. Repeat the example in Section 4.1 using $x(t) = u(t)\exp(-\alpha t)$.

4.4. Prove the properties of the radar ambiguity function.

4.5. A radar system uses LFM waveforms. The received signal is of the form $s_r(t) = As(t-\tau)+n(t)$, where τ is a time delay that depends on range, $s(t) = Rect(t/\tau')\cos(2\pi f_0 t - \phi(t))$, and $\phi(t) = -\pi B t^2/\tau'$. Assume that the radar bandwidth is $B = 5MHz$, and the pulse width is $\tau' = 5\mu s$. (a) Give the quadrature components of the matched filter response that is matched to $s(t)$. (b) Write an expression for the output of the matched filter. (c) Compute the increase in SNR produced by the matched filter.

4.6. (a) Write an expression for the ambiguity function of an LFM waveform, where $\tau' = 6.4\mu s$ and the compression ratio is 32. (b) Give an expression for the matched filter impulse response.

4.7. (a) Write an expression for the ambiguity function of a LFM signal with bandwidth $B = 10MHz$, pulse width $\tau' = 1\mu s$, and wavelength $\lambda = 1cm$. (b) Plot the zero Doppler cut of the ambiguity function. (c) Assume a target moving toward the radar with radial velocity $v_r = 100m/s$. What is the Doppler shift associated with this target? (d) Plot the ambiguity function for the Doppler cut in part (c). (e) Assume that three pulses are transmitted with PRF $f_r = 2000Hz$. Repeat part (b).

4.8. (a) Give an expression for the ambiguity function for a pulse train consisting of 4 pulses, where the pulse width is $\tau' = 1\mu s$ and the pulse repetition interval is $T = 10\mu s$. Assume a wavelength of $\lambda = 1cm$. (b) Sketch the ambiguity function contour.

4.9. Hyperbolic frequency modulation (HFM) is better than LFM for high radial velocities. The HFM phase is

$$\phi_h(t) = \frac{\omega_0^2}{\mu_h}\ln\left(1 + \frac{\mu_h \alpha t}{\omega_0}\right)$$

where μ_h is an HFM coefficient and α is a constant. (a) Give an expression for the instantaneous frequency of an HFM pulse of duration τ'_h. (b) Show that HFM can be approximated by LFM. Express the LFM coefficient μ_l in terms of μ_h and in terms of B and τ'.

4.10. Consider a sonar system with range resolution $\Delta R = 4cm$. (a) A sinusoidal pulse at frequency $f_0 = 100KHz$ is transmitted. What is the pulse width, and what is the bandwidth? (b) By using an up-chirp LFM, centered at f_0, one can increase the pulse width for the same range resolution. If you want to increase the transmitted energy by a factor of 20, give an expression for the transmitted pulse. (c) Give an expression for the causal filter matched to the LFM pulse in part b.

4.11. A pulse train $y(t)$ is given by

$$y(t) = \sum_{n=0}^{2} w(n)x(t - n\tau')$$

where $x(t) = \exp(-t^2/2)$ is a single pulse of duration τ' and the weighting sequence is $\{w(n)\} = \{0.5, 1, 0.7\}$. Find and sketch the correlations R_x, R_w, and R_y.

4.12. Repeat the previous problem for $x(t) = \exp(-t^2/2)\cos 2\pi f_0 t$.

4.13. Derive Eq. (4.29) and Eq. (4.30) when the input noise is not white.

4.14. Show that the zero Doppler cut for the ambiguity function of an arbitrary phase coded pulse with a pulse width τ_p is given by $Y(f) = |\sin c(f\tau_p)|^2$.

4.15. Show that

$$\int_{-\infty}^{\infty} t x^*(t) x'(t) \; dt = -\int_{-\infty}^{\infty} f X^*(f) X'(f) \; df$$

where $X(f)$, is the FT of $x(t)$ and $x'(t)$ is its derivative with respect to time. The function $X'(f)$ is the derivative of $X(f)$ with respect to frequency.