第四章 $De\ Giorgi$ 迭代和 $Moser$ 迭代技术

张祖锦(David Zhang)
中山大学数计学院

1. 设 $p>n\geq 3$，$0\leq c\leq M$，$f\in L^{p_*}(\Omega)$，$f^i\in L^p(\Omega)$，$u\in H^1(\Omega)$ 为方程

$$(E1)-\Delta u+cu=f+D_if_i,x\in\Omega$$

的弱解,则

$$\sup_{\Omega}u\leq\sup_{\partial\Omega}u_++C\left[\|f\|_{L^{p*}}+\left\|f^i\right\|_{L^p}\right]|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

$$\inf_{\Omega}u\leq\inf_{\partial\Omega}u_--C\left[\|f\|_{L^{p*}}+\left\|f^i\right\|_{L^p}\right]|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

其中 $p_*=\dfrac{np}{n+p}$，$s_-=\min\{s,0\}$，$C=C(n,p,M,\Omega)$ 且与 $|\Omega|$ 的下界无关.

证明 (a) 设 $l=\sup_{\partial\Omega}u_+$.对 $\forall k>l$，作 $\varphi_k=(u-k)_+\in H_0^1(\Omega)$，作为试验函数代入方程 $(E1)$ 弱解的定义式,有

$$\int_\Omega\nabla u\bullet\nabla\varphi_kdx+\int_\Omega cu\varphi_kdx=\int_\Omega\left(f\varphi_k+f^iD_i\varphi_k\right)dx$$

即

$$\int_\Omega\left|\nabla\varphi_k\right|^2dx+\int_\Omega c\varphi_k^2dx+k\int_\Omega\varphi_kdx=\int_\Omega f\varphi_kdx+\int_\Omega f^iD_i\varphi_kdx$$

记 $A(k)=\{x\in\Omega;u(x)\geq k\}$,则

$$\int_{A(k)}\left|\nabla\varphi_k\right|^2dx+\int_{A(k)}c\varphi_k^2dx+k\int_{A(k)}\varphi_kdx=\int_{A(k)}f\varphi_kdx+\int_{A(k)}f^iD_i\varphi_kdx\,(*)$$

现如今,$0\leq c\leq M,k>l=\sup_{\partial\Omega}u_+\geq 0,\varphi_k=(u-k)_+\geq 0$,有

$(*)$ 式左端 $\geq\int_{A(k)}\left|\nabla\varphi_k\right|^2dx$;

对 $(*)$ 右端第一式,因 $f\in L^{p_*}$，$\varphi_k\in H_0^1(\Omega)\subset L^{2^*}$,利用推广的 $Holder$ 不等式,有

$$\frac{n+p}{np}+\frac{n-2}{2n}+\left(\frac{1}{2}-\frac{1}{p}\right)=1$$

$$\int_{A(k)}f\varphi_kdx\leq\|f\|_{L^{p*}}\left\|\varphi_k\right\|_{L^{2^*}}|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

对 $(*)$ 右端第二式,因 $f^i\in L^p,D_i\varphi_k\in L^2$,同样利用推广的 $Holder$ 不等式,有

$$\frac{1}{p}+\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{p}\right)=1$$

$$\int_{A(k)}f^iD_i\varphi_kdx\leq\left\|f^i\right\|_{L^p}\left\|\nabla\varphi_k\right\|_{L^2}|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

于是我们有

$$\int_{A(k)}\left|\nabla\varphi_k\right|^2dx\leq\left[\|f\|_{L^{p*}}\left\|\varphi_k\right\|_{L^2(A(k))}+\left\|f^i\right\|_{L^p}\left\|\nabla\varphi_k\right\|_{L^2(A(k))}\right]|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

再利用 $Poincare$ 不等式及带 ε 的 $Cauchy$ 不等式有

$$\left\|\nabla\varphi_k\right\|_{L^2(A(k))}\leq CF|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

其中 $F=\|f\|_{L^{p*}}+\left\|f^i\right\|_{L^p}$.

又$\forall h>k$,由$Sobolev$嵌入定理,有

$$(h-k)|A(h)|^{\frac{1}{2^*}}\le\|\varphi_k\|_{L^{2^*}(A(h))}\le\|\varphi_k\|_{L^{2^*}(A(k))}\le\|\nabla\varphi_k\|_{L^2(A(k))}\le CF|A(k)|^{\frac{1}{2}-\frac{1}{p}}$$

从而

$$|A(h)|\le\left(\frac{CF}{h-k}\right)^{2^*}|A(k)|^{\frac{n(p-2)}{p(n-2)}}$$

由$\alpha=2^*=\dfrac{2n}{n-2}>0,\beta=\dfrac{n(p-2)}{p(n-2)}>1$及$De\ Giorgi$迭代引理,我们有

$$A(l+d)=0$$

其中$d=CF|A(l)|^{\frac{\beta-1}{\alpha}}2^{\frac{\beta}{\beta-1}}$.

由$A(k)$的定义即有

$$u(x)\le l+d\le l+CF|\Omega|^{\frac{1}{n}-\frac{1}{p}},a.e.$$

$$\sup_{\Omega}u\le\sup_{\partial\Omega}u_++C\left[\|f\|_{L^{p*}}+\|f^i\|_{L^p}\right]|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

(b)

$$\inf_{\Omega}u=-\sup_{\Omega}(-u)\ge-\sup_{\partial\Omega}(-u)_+-C\left[\|f\|_{L^{p*}}+\|f^i\|_{L^p}\right]|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

$$=\inf_{\partial\Omega}u_--C\left[\|f\|_{L^{p*}}+\|f^i\|_{L^p}\right]|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

最后一步是因为

$$u_-=min\{u,0\}=\begin{cases}0, & u>0;\\ u & u\le 0.\end{cases}=\begin{cases}-(-u), & -u\ge 0;\\ 0, & -u<0.\end{cases}$$

$$=-\begin{cases}-u, & -u\ge 0;\\ 0, & -u<0.\end{cases}=-(-u)_+$$

及$\inf A=-\sup(-A)$

注 推广的$Holder$不等式:$\forall n\in Z_+$,若$1\le p_i\le\infty$,$\sum_{i=1}^{n}\dfrac{1}{p_i}=1$,$f_i\in L^{p_i}$,则

$$\int\prod_{i=1}^{n}f_i\le\prod_{i=1}^{n}\|f_i\|_{p_i}$$

证明 用数学归纳法.

初始步 $k=1$明显;$k=2$即为通常$Holder$不等式;

递归步 假设对$k\le n$不等式成立,当$k=n+1$时,

$$\int\prod_{i=1}^{n+1}f_i=\int\left(\prod_{i=1}^{n}f_i\right)f_{n+1}\le\left[\int\prod_{i=1}^{n}f_i^{\frac{p_{n+1}}{p_{n+1}-1}}\right]^{\frac{p_{n+1}-1}{p_{n+1}}}\|f_{n+1}\|_{p_{n+1}}$$

$$\le\prod_{i=1}^{n}\left(\int|f_i|^{\frac{p_{n+1}}{p_{n+1}-1}\frac{p_i(p_{n+1}-1)}{p_{n+1}}}\right)^{\frac{p_{n+1}}{p_i(p_{n+1}-1)}\frac{p_{n+1}-1}{p_{n+1}}}\|f_{n+1}\|_{p_{n+1}}$$

$$=\prod_{i=1}^{n}\|f_i\|_{p_i}\|f_{n+1}\|_{p_{n+1}}=\prod_{i=1}^{n+1}\|f_i\|_{p_i}$$

2. $p>n\geq 3; 0\leq c\leq M; f\in L^\infty\left(0,T;L^{p_*}(\Omega)\right), f^i\in L^\infty\left(0,T;L^p(\Omega)\right), i=1,2,\cdots,n$

$u\in W_2^{1,1}(Q_T)$为非齐次热方程

$$(E2)\, u_t-\Delta u+cu=f+D_i f^i, (x,t)\in Q_T$$

的弱解,则

$$\sup_{Q_T} u\leq \sup_{\partial_p Q_T} u_+ + CF|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

$$\inf_{Q_T} u\geq \inf_{\partial_p Q_T} u_- - CF|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

其中$p_*=\dfrac{np}{n+p}, C=C(n,p,M,\Omega)$与$|\Omega|$的下界无关,

$$F=\sup_{0<t<T}\|f\|_{L^{p_*}}+\sup_{0<t<T}\left\|f^i\right\|_{L^p}$$

证明:(a) 设$l=\sup_{\partial_p Q_T} u_+$. $\forall k>l$,记$\varphi=(u-k)_+\chi_{[t_1,t_2]}\in \overset{\bullet}{W}{}_2^{1,1}(Q_T)\subset \overset{o}{W}{}_2^{1,0}(Q_T)$,其中$\chi_{[t_1,t_2]}$为区间$[t_1,t_2]$的特征函数,$0\leq t_1<t_2\leq T$,作为试验函数代入非齐次热方程$(E2)$弱解的定义式,有

$$\iint_{Q_T}\left(u_t\varphi+\nabla u\bullet\nabla\varphi+cu\varphi\right)dxdt=\iint_{Q_T}\left(f\varphi-f^iD_i\varphi\right)dxdt$$

即

$$\iint_{Q_T}\left(u_t\varphi+|\nabla\varphi|^2+c\varphi^2\right)dxdt+k\iint_{Q_T}c\varphi dxdt=\iint_{Q_T}\left(f\varphi-f^iD_i\varphi\right)dxdt$$

又$0\leq c\leq M, k>l=\sup_{\partial_p Q_T} u_+\geq 0$,有

$$\iint_{Q_T}\left(u_t\varphi+|\nabla\varphi|^2+c\varphi^2\right)dxdt\leq\iint_{Q_T}\left(f\varphi-f^iD_i\varphi\right)dxdt$$

现如今,取

$$I_k(t)=\int_\Omega (u-k)_+^2\,dx$$

则I_k为t之绝对连续函数,设其在$t=\sigma$处取得最大值,因$I_k(0)=0, I_k(t)\geq 0$,不妨设$\sigma>0$,对充分小$\varepsilon>0$,有

$$\int_\Omega\varphi^2dx\,\Big|_{t=\sigma-\varepsilon}^{t=\sigma}+\int_{\sigma-\varepsilon}^{\sigma}\int_\Omega|\nabla\varphi|^2dxdt\leq\int_{\sigma-\varepsilon}^{\sigma}\int_\Omega f\varphi dxdt-\int_{\sigma-\varepsilon}^{\sigma}\int_\Omega f^iD_i\varphi dxdt$$

将第一项省去,两边同时除以ε后令$\varepsilon\to 0$,有

$$\int_\Omega|\nabla\varphi(x,\sigma)|^2dx\leq\int_\Omega|f(x,\sigma)\varphi(x,\sigma)|dx+\int_\Omega\left|f^i(x,\sigma)D_i\varphi(x,\sigma)\right|dx$$

记

$$A_k(t)=\left\{x\in\Omega; u(x,t)>k\right\}, u_k=\sup_{0<t<T}|A_k(t)|,$$

又有

$$\int_{A_k(\sigma)}|\nabla\varphi(x,\sigma)|^2dx\leq\int_{A_k(\sigma)}|f(x,\sigma)\varphi(x,\sigma)|dx+\int_{A_k(\sigma)}\left|f^i(x,\sigma)D_i\varphi(x,\sigma)\right|dx$$

对上式右端,同题 1,利用推广的$Holder$不等式,有

$$\frac{n+p}{np}+\frac{n-2}{2n}+\left(\frac{1}{2}-\frac{1}{p}\right)=1$$

$$\int_{A_k(\sigma)}\left|f(x,\sigma)\varphi(x,\sigma)\right|dx\le\left\|f(x,\sigma)\right\|_{L^{p*}}\left\|\varphi(x,\sigma)\right\|_{L^{2^*}}\left|A_k(\sigma)\right|^{\frac12-\frac1p}$$

$$\le\sup_{0<t<T}\left\|f\right\|_{L^{p*}}\left\|\varphi(x,\sigma)\right\|_{L^{2^*}}\left|A_k(\sigma)\right|^{\frac12-\frac1p}$$

$$\frac1p+\frac12+\left(\frac12-\frac1p\right)=1$$

$$\int_{A_k(\sigma)}\left|f^i(x,\sigma)D_i\varphi(x,\sigma)\right|dx\le\left\|f^i(x,\sigma)\right\|_{L^p}\left\|\nabla\varphi(x,\sigma)\right\|_{L^2}\left|A_k(\sigma)\right|^{\frac12-\frac1p}$$

$$\le\sup_{0<t<T}\left\|f^i\right\|_{L^p}\left\|\nabla\varphi(x,\sigma)\right\|_{L^2}\left|A_k(\sigma)\right|^{\frac12-\frac1p}$$

于是由$Sobolev$嵌入定理及带ε的$Cauchy$不等式,有

$$\left(\int_{A_k(\sigma)}\left|\varphi(x,\sigma)\right|^{2^*}dx\right)^{\frac{1}{2^*}}\le C\left(\int_{A_k(\sigma)}\left|\nabla\varphi(x,\sigma)\right|^2dx\right)^{\frac12}\le CF\left|A_k(\sigma)\right|^{\frac12-\frac1p}$$

好了, 开始迭代了…$\forall h>k$,有

$$t\in[0,T],x\in A_k(t)\Rightarrow\varphi(x,t)=(u-k)_+\ge h-k$$

从而

$$(h-k)^2\left|A_h(t)\right|\le\int_{A_h(t)}\varphi^2(x,t)dx\le\int_{A_k(t)}\varphi^2(x,t)dx=I_k(t)\le I_k(\sigma)$$

$$=\int_{A_k(\sigma)}\varphi^2(x,\sigma)dx\le\left(\int_{A_k(\sigma)}\varphi^{2\frac{2^*}{2}}(x)dx\right)^{\frac{2}{2^*}}\left|A_k(\sigma)\right|^{\frac{2^*-2}{2^*}}$$

$$\le(CF)^2\left|A_k(\sigma)\right|^{2\left(\frac12-\frac1p\right)+\frac{2^*-2}{2^*}}=(CF)^2\left|A_k(\sigma)\right|^{1-\frac2p+\frac2n}\le(CF)^2u_k^{1-\frac2p+\frac2n}$$

由t的任意性,有

$$u_h\le\left(\frac{CF}{h-k}\right)^2u_k^{1-\frac2p+\frac2n}$$

由$\alpha=2>0,\beta=1-\dfrac2p+\dfrac2n>1$及$De\ Giorgi$迭代引理,有

$$u_{l+d}=0$$

其中$d=CFu_l^{\frac{\beta-1}{\alpha}}2^{\frac{\alpha}{\beta-1}}$,于是由$u_k$的定义,

$$u(x,t)\le l+CFu_l^{\frac1n-\frac1p}\le l+CF\left|\Omega\right|^{\frac1n-\frac1p},a.e.\text{于}Q_T$$

(b) $\inf_{Q_T}u=-\sup_{Q_T}(-u)\ge-\sup_{\partial_pQ_T}(-u)_+-CF\left|\Omega\right|^{\frac1n-\frac1p}=\inf_{\partial_pQ_T}u_--CF\left|\Omega\right|^{\frac1n-\frac1p}$

3. 设$x^0\in\Omega$,$B_R=B_R\left(x^0\right)\subset\Omega$,$u\in H^1(\Omega)\bigcap L^\infty(\Omega)$为$Laplace$方程

$$(E3)-\Delta u=0,x\in\Omega$$

的弱下解,则$\forall\theta\in(0,1),\exists C=C(n,\theta),s.t.$

$$\sup_{B_{\theta R}}u\le C\left(\frac{1}{R^n}\int_{B_R}u_+^2dx\right)^{\frac12}$$

证明 分七步走完.

第一步 化简

由于 $g(s)=s_+$ 是凸的,单调不减的,$Lipschitz$ 连续的,我们有 u_+ 也是方程 $(E3)$ 的弱下解.若对其有所证,更有对 u 的所证.故如今只考虑 $u\geq 0$ 情形.

第二步 证明反向 $Poincare$ 不等式

$$p\geq 2\Rightarrow\int_{B_R}\eta^2\left|\nabla u^{p/2}\right|^2dx\leq C\int_{B_R}u^p\left|\nabla\eta\right|^2dx$$

其中 η 为 $B_{\rho'}$ 相对于 $B_\rho\ (0<\rho<\rho'\leq R)$ 的截断函数在积分区域 B_R 上的零延拓,即满足

$$\eta(x)\in C_0^\infty\left(B_{\rho'}\right);0\leq\eta(x)\leq 1\text{于}B_\rho;\eta(x)=0\text{于}\Omega\setminus B_{\rho'};\left|\nabla\eta(x)\right|\leq\frac{C}{\rho'-\rho}.$$

取 η^2u^{p-1} 作为试验函数代入 $(E3)$ 弱下解的定义式,有

$$\begin{aligned}
&0\geq\int_{B_R}\nabla u\bullet\nabla\left(\eta^2u^{p-1}\right)dx\\
&=2\int_{B_R}\eta u^{p-1}\nabla u\bullet\nabla\eta dx+(p-1)\int_{B_R}\eta^2u^{p-2}\left|\nabla u\right|^2dx\\
&=\frac{4}{p}\int_{B_R}\eta u^{p/2}\nabla u^{p/2}\bullet\nabla\eta dx+\frac{4(p-1)}{p^2}\int_{B_R}\eta^2\left|\nabla u^{p/2}\right|^2dx\\
&\geq\frac{4}{p}\left[\frac{p-1}{p}\int_{B_R}\eta^2\left|\nabla u^{p/2}\right|^2dx-\frac{\varepsilon}{2}\int_{B_R}\eta^2\left|\nabla u^{p/2}\right|^2dx-\frac{1}{2\varepsilon}\int_{B_R}u^p\left|\nabla\eta\right|^2dx\right]\\
&=\frac{4}{p}\left[\left(\frac{p-1}{p}-\frac{\varepsilon}{2}\right)\int_{B_R}\eta^2\left|\nabla u^{p/2}\right|^2dx-\frac{1}{2\varepsilon}\int_{B_R}u^p\left|\nabla\eta\right|^2dx\right]
\end{aligned}$$

取 $\varepsilon\in\left(0,\frac{2(p-1)}{p}\right)$,就有所证.

第三步 证明球域上的推广 $Poincare$ 不等式

若 $u\in W_0^{1,p}\left(B_R\right)$,$1\leq p<n$,则 $\forall 1\leq q\leq p^*$,$\exists C=C(n,p),s.t.$

$$\left(\int_{B_R}|u|^q\,dx\right)^{\frac{1}{q}}\leq CR^{1+\frac{n}{q}-\frac{n}{p}}\left(\int_{B_R}|Du|^p\,dx\right)^{\frac{1}{p}}$$

(a) $R=1$ 之情形. 由 $W_0^{1,p}\left(B_R\right)\subset L^q\left(B_R\right)$ 及 $Poincare$ 不等式,有

$$\left(\int_{B_1}|u|^q\,dx\right)^{\frac{1}{q}}\leq C\|u\|_{W^{1,p}(B_1)}\leq C\left\|Du\right\|_{L^p(B_1)}$$

(b) 一般情形. 利用 $Rescaling$ 技术,令 $y=\frac{x}{R}$,有

$$\begin{aligned}
&\left(\int_{B_R}|u(x)|^q\,dx\right)^{\frac{1}{q}}=\left(\int_{B_1}\left|u(y)\right|^qR^ndy\right)^{\frac{1}{q}}\leq CR^{\frac{n}{q}}\left(\int_{B_1}\left|Du(y)\right|^pdy\right)^{\frac{1}{p}}\\
&=CR^{\frac{n}{q}}\left(\int_{B_R}\left|Du(x)\right|R^p\frac{1}{R^n}dx\right)^{\frac{1}{p}}=CR^{1+\frac{n}{q}-\frac{n}{q}}\left(\int_{B_R}\left|Du(x)\right|^pdx\right)^{\frac{1}{p}}
\end{aligned}$$

第四步 证明反向 $Holder$ 不等式

$$p\geq 2\Rightarrow\left(\frac{1}{R^n}\int_{B_\rho}u^{pq}dx\right)^{\frac{1}{q}}\leq C\left(\frac{1}{R^{n-2}(\rho'-\rho)^2}\int_{B_{\rho'}}u^pdx\right)$$

其中

$$1<q<\begin{cases}+\infty, & n=1,2;\\ \dfrac{n}{n-2}, & n>2.\end{cases}$$

因当 $n>2$ 时,$2q<\dfrac{2n}{n-2}=2^*$,在第三步的证明中令 u 为 η^2u^p,有

$$\left(\frac{1}{R^n}\int_{B_R}\eta^{2q}u^{pq}dx\right)^{\frac{1}{2q}}=R^{-\frac{n}{2q}}\left(\int_{B_R}\left(\eta u^{p/2}\right)^{2q}dx\right)^{\frac{1}{2q}}$$

$$\le CR^{-\frac{n}{2q}}R^{1+\frac{n}{2q}-\frac{n}{2}}\left(\int_{B_R}\left|\nabla\left(\eta u^{p/2}\right)\right|^2dx\right)^{\frac{1}{2}}$$

$$\le C\left(\frac{1}{R^{n-2}}\int_{B_R}u^p\left|\nabla\eta\right|^2+\eta^2\left|\nabla u^{p/2}\right|^2dx\right)^{\frac{1}{2}}$$

$$\le C\left(\frac{1}{R^{n-2}}\int_{B_R}u^p\left|\nabla\eta\right|^2dx\right)^{\frac{1}{2}}\le C\left(\frac{1}{R^{n-2}\left(\rho'-\rho\right)^2}\int_{B_{\rho'}}u^pdx\right)^{\frac{1}{2}}$$

其中第二个不等号用到第一步证的反向 $Poincare$ 不等式.

两边平方即为所证.

第五步 迭代

设 $\rho_k=R\left[\theta+(1-\theta)\theta^k\right],k=0,1,2,\cdots$,则 $\rho_0=R$,$\lim\limits_{k\to\infty}\rho_k=\theta R$,且 $\{\rho_k\}$ 递减.由第四步所证反向 $Holder$ 不等式中令 $p=2q^k$,$\rho=\rho_{k+1},\rho'=\rho_k$,得

$$a_{k+1}\equiv\left(\frac{1}{R^n}\int_{B_{\rho_{k+1}}}u^{2q^{k+1}}dx\right)^{\frac{1}{2q^{k+1}}}$$

$$\le C^{\frac{1}{2q^k}}\theta^{\frac{k}{q^k}}(1-\theta)^{\frac{2}{q^k}}\left(\frac{1}{R^n}\int_{B_{\rho_k}}u^{2q^k}dx\right)^{\frac{1}{2q^k}}=C^{\frac{1}{2q^k}}\theta^{\frac{k}{q^k}}(1-\theta)^{\frac{2}{q^k}}a_k$$

反复迭代,有 $a_k\le C^\alpha\theta^\beta(1-\theta)^\gamma a_0$,其中

$$\alpha=\sum_{k=0}^{\infty}\frac{1}{2q^k},\beta=\sum_{k=0}^{\infty}\frac{k}{q^k},\gamma=\sum_{k=0}^{\infty}\frac{2}{q^k}[\gamma=4\alpha]$$

由于 $q>1$,上述三个技术均收敛,α,β,γ 均为有限实数,而有 $a_k\le Ca_0$.

第六步 证明 L^p 与 L^∞ 之关系

设 $\Omega\subset R^n$ 是有界可测集,u 于 Ω 上可测,且存在一递增自然数列 $\{p_k\}_{k=1}^{\infty}$ 使得

$(i)\lim\limits_{k\to\infty}p_k=\infty;$

$(ii)u\in L^{p_k}(\Omega)$,且 $\exists C>0,s.t.\|u\|_{L^{p_n}}\le C$[即数列 $\{\|u\|_{L^{p_n}}\}_{n=1}^{\infty}$ 有界]

则 $(i)u\in L^\infty(\Omega);(ii)\|u\|_{L^\infty}\le C$

$(a)u\in L^\infty(\Omega)$

用反证法. 若不然,$A=\{x\in\Omega;u(x)>C+1\}$ 有正测度,而有

$$\|u\|_{L^{p_n}}\ge\|u\|_{L^{p_n}(A)}\ge(C+1)|A|^{\frac{1}{p_n}}\to C+1>C\,(n\to\infty)$$

矛盾.

$(b)\|u\|_{L^\infty}\le C$

这是因为$\|u\|_{L^\infty} \leftarrow \|u\|_{L^{p_n}} \le C$.

第七步 证明题目

由第五步,$u \in L^{2q^k}(B_{\theta R})$且$\|u\|_{L^{2q^k}(B_{\theta R})} \le Ca_0$,于是由第六步知

$$(ess)\sup_{B_{\theta R}} u = \|u\|_{L^\infty(B_{\theta R})} \le Ca_0 = C\left(\frac{1}{R^n}\int_{B_R} u^2 dx\right)^{\frac{1}{2}}$$

注 曹植之七步诗

煮豆燃豆萁,豆在釜中泣.
本是同根生,相煎何太急?

4. 设$0 \le c \le M, f \in L^\infty(M), u \in H^1(\Omega)\bigcap L^\infty(\Omega)$为方程

$$(E4) -\Delta u + cu = f, x \in \Omega$$

的弱下解,则$\exists R_0 = R_0(M) > 0, s.t. \forall R \in (0, R_0], \forall x^0 \in \Omega, B_R = B_R(x^0) \subset \Omega, \forall \theta \in (0,1)\ \exists C = C(n, R_0, M, \theta) > 0, s.t.$

$$\sup_{B_{\theta R}} u \le C\left(\frac{1}{R^n}\int_{B_R} u^2 dx\right)^{\frac{1}{2}} + C\|f\|_\infty$$

证明 $\forall x^0 \in \Omega$,作$u^- = u + |x - x^0|^2 \|f\|_\infty$,则对$\forall 0 \le \varphi \in C_0^\infty(\Omega)$,都有

$$\begin{aligned}
&\int_\Omega \nabla u^- \bullet \nabla\varphi + cu^-\varphi dx = \\
&\int_\Omega \nabla u \bullet \nabla\varphi + cu\varphi dx + \|f\|_\infty \int_\Omega \nabla|x - x^0|^2 \bullet \nabla\varphi + c|x - x^0|^2 \varphi dx \\
&\le \int_\Omega f\varphi dx + \|f\|_\infty \int_\Omega -2n\varphi + M|x - x^0|^2 \varphi dx \\
&\le \|f\|_\infty \int_\Omega \varphi\left[1 - 2n + M|x - x^0|^2\right] dx
\end{aligned}$$

取$R_0^2 \in \left(0, \frac{1}{M}\right)$,则有$1 - 2n + M|x - x^0|^2 < 1 - 2n + 1 = 2(1-n) \le 0$,又$c \ge 0$,有

$$\int_\Omega \nabla u^- \bullet \nabla\varphi dx \le -\int_\Omega cu^-\varphi dx \le 0, \int_\Omega \nabla u_+^- \bullet \nabla\varphi dx \le 0$$

u_+^-为$(E3)$的弱下解,由题 3 我们有

$$\sup_{B_{\theta R}} u_+^- \le C\left(\frac{1}{R^n}\int_{B_R} (u_+^-)^2 dx\right)^{\frac{1}{2}}$$

而[表递进]

$$\begin{aligned}
&\sup_{B_{\theta R}} u = \sup_{B_{\theta R}}\left[u^- - |x - x^0|^2 \|f\|_\infty\right] \le \sup_{B_{\theta R}} u^- + \|f\| \sup_{B_{\theta R}}\left[-|x - x^0|^2\right] \\
&= \sup_{B_{\theta R}} u_+^- + C\|f\|_\infty \le C\left(\frac{1}{R^n}\int_{B_R} (u_+^-)^2 dx\right)^{\frac{1}{2}} + C\|f\|_\infty \\
&\le C\left(\frac{1}{R^n}\int_{B_R} u^2 dx\right)^{\frac{1}{2}} + C\|f\|_\infty
\end{aligned}$$