第二章 线性椭圆方程的L^2理论

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1 设Ω是$\mathbb{R}^n$中的有界区域,$\partial\Omega$充分光滑,$f \in L^2(\Omega), g \in H^1(\Omega)$.考虑边值问题:

$$(D1) \begin{cases} -\triangle u(x) = f(x), x \in \Omega \\ \gamma(u-g) = 0 \end{cases}$$

这里γ表示∂D上的迹算子,即从$C^\infty(\bar{\Omega})$到$L^2(\partial\Omega)$的映射$\gamma : w \mapsto w|_{\partial\Omega}(\forall w \in C^\infty(\bar{\Omega}))$到$H^1(\Omega)$的连续延拓.令$J$是$H^1(\Omega)$上的下列泛函：

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 dx + \int_\Omega \nabla g(x) \cdot \nabla w(x) dx - \int_\Omega f(x) w(x) dx, \forall w \in H_0^1(\Omega)$$

1) 对$u \in H^1(\Omega)$给出u是问题$(D1)$的弱解的定义.
2) $u \in H^1(\Omega)$是问题$(D1)$的弱解当且仅当$w = u - g \in H_0^1(\Omega)$是$w$的泛函$J$的最小元, 即成立$J(w) \leq J(\varphi), \forall \varphi \in H_0^1(\Omega)$
3) 叙述泛函弱下半连续的定义和与之相关的变分原理.
4) 根据2), 3),证明:问题$(D1)$存在唯一的弱解.

证明：

1) 若对$\forall \psi \in C_0^\infty(\Omega)$,从而对$\forall \psi \in H_0^1(\Omega)$,都有

$$\int_\Omega \nabla u \cdot \nabla \psi dx = \int_\Omega f \cdot \psi dx$$

且$\gamma(u-g) = 0$,则称u使问题$(D1)$的弱解.

2) $i)$ 当且(充分性)
$\forall \psi \in H_0^1(\Omega)$,作函数

$$\begin{array}{rcl} F : \mathbb{R} & \to & \mathbb{R} \\ \varepsilon & \mapsto & J(w + \varepsilon\psi) \end{array}$$

则

$$\begin{aligned} F(\varepsilon) &= J(w + \varepsilon\psi) \\ &= \frac{1}{2}\int_\Omega |\nabla(w + \varepsilon\psi)|^2 dx + \int_\Omega \nabla g \cdot \nabla(w + \varepsilon\psi) dx - \int_\Omega f(w + \varepsilon\psi) dx \\ &= \frac{1}{2}\int_\Omega |\nabla w|^2 dx + \varepsilon\int_\Omega \nabla w \cdot \nabla\psi dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\psi|^2 dx \\ &\quad + \int_\Omega \nabla g \cdot \nabla w dx + \varepsilon\int_\Omega \nabla g \cdot \nabla\psi dx - \int_\Omega f w dx - \varepsilon\int_\Omega f\psi dx \end{aligned}$$

F是可微函数,且在$\varepsilon = 0$处取得最小值,从而由$Fermat$定理,有

$$F'(0) = \int_\Omega \nabla w \cdot \nabla\psi dx + \int_\Omega \nabla g \cdot \nabla\psi dx - \int_\Omega f\psi dx = 0$$

此即

$$\int_\Omega \nabla u \cdot \nabla\psi dx = \int_\Omega f\psi dx$$

于是:

a) 对$\forall \psi \in H_0^1(\Omega)$,都有

$$\int_\Omega \nabla u \cdot \nabla\psi dx = \int_\Omega f \cdot \psi dx$$

b)

$$\gamma(u-g) = \gamma w = 0$$

$ii)$ 仅当(必要性)

$\forall\varphi\in H_0^1(\Omega)$,有$\varphi-w\in H_0^1(\Omega)$,作为试验函数代入弱解的定义式,有

$$\int_\Omega \nabla u\cdot\nabla(\varphi-w)dx=\int_\Omega f(\varphi-w)dx$$

即

$$\int_\Omega \nabla g\cdot\nabla(\varphi-w)dx-\int_\Omega f(\varphi-w)dx=-\int_\Omega \nabla w\cdot\nabla(\varphi-w)dx$$

而有

$$\begin{aligned}J(\varphi)-J(w) &= \frac{1}{2}\int_\Omega |\nabla\varphi|^2-|\nabla w|^2dx+\int_\Omega g\cdot\nabla(\varphi-w)dx-\int_\Omega f(\varphi-w)dx\\ &= \frac{1}{2}\int_\Omega |\nabla\varphi|^2-|\nabla w|^2dx\int_\Omega \nabla w\cdot\nabla(\varphi-w)dx\\ &= \frac{1}{2}\int_\Omega |\nabla(\varphi-w)|^2dx\geq 0\end{aligned}$$

从而得证.

3) $i)$ 泛函弱下半连续的定义:

设X是一B^*空间,f是X上的泛函.若

$$\forall\{x_n\}\subset X,x_n\rightharpoonup x\Rightarrow f(x)\leq\liminf_{n\to\infty}f(x_n)$$

则称f是弱下半连续的.

$ii)$ 变分原理:

对$J:H_0^1(\Omega)\to\mathbb{R}$,我们有

a) J是弱下半连续的.

对

$$\forall\{\varphi_n\}\subset H_0^1(\Omega),\varphi_n\rightharpoonup\varphi$$

设

$$J(\varphi)=J_1(\varphi)+J_2(\varphi)$$

其中

$$\begin{aligned}J_1(\varphi) &= \frac{1}{2}\int_\Omega|\nabla\varphi|^2dx\\ J_2(\varphi) &= \int_\Omega\nabla g\cdot\nabla\varphi dx-\int_\Omega f\varphi dx,\forall\varphi\in H_0^1(\Omega)\end{aligned}$$

对J_2,我们有其为$H_0^1(\Omega)$上的有界线性泛函:

一方面,

$$\begin{aligned}J_2(\lambda_i\varphi_i) &= \int_\Omega\nabla g\cdot\nabla(\lambda_i\varphi_i)dx-\int_\Omega f(\lambda_i\varphi_i)dx\\ &= \lambda_i\left(\int_\Omega\nabla g\cdot\nabla\varphi_i dx-\int_\Omega f\varphi_i dx\right)\\ &= \lambda_i J_2(\varphi_i)\end{aligned}$$

另一方面,

$$\begin{aligned}|J_2(\varphi)| &\leq \left(\int_\Omega|\nabla g|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla\varphi|^2dx\right)^{\frac{1}{2}}+\left(\int_\Omega|f|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega|\varphi|^2dx\right)^{\frac{1}{2}}\\ &\leq 2max\left\{||\nabla g||_{L^2(\Omega)},||f||_{L^2(\Omega)}\right\}||\varphi||_{H^1(\Omega)}\end{aligned}$$

从而由弱收敛的定义,有

$$J_2(\varphi_n)\to J_2(\varphi)$$

对J_1,我们设$\psi_n=\varphi_n-\varphi$,则$\psi_n\rightharpoonup 0$,从而

$$\begin{aligned}
J_1(\varphi_n) &= J_1(\psi_n+\varphi)\\
&= \frac{1}{2}\int_\Omega |\nabla(\psi_n+w)|^2dx\\
&= J(\psi_n)+\int_\Omega \nabla\psi_n\cdot\nabla\varphi dx+J(\varphi)\\
&\geq \int_\Omega \nabla\psi_n\cdot\nabla\varphi dx+J(\varphi)
\end{aligned}$$

两边取下极限,有

$$\liminf_{n\to\infty} J_1(\varphi_n)\geq \lim_{n\to\infty}\int_\Omega \nabla\psi_n\cdot\nabla\varphi dx+J(\varphi)=J(\varphi)$$

综合对J_1,J_2的讨论,我们有J是弱下半连续的.

b) J有下界.

$$\begin{aligned}
J(\varphi) &= \frac{1}{2}\int_\Omega |\nabla\varphi|^2dx+\int_\Omega \nabla g\cdot\nabla\varphi dx-\int_\Omega f\varphi dx\\
&\geq \frac{1-\varepsilon}{2}\int_\Omega |\nabla\varphi|^2dx-\frac{\varepsilon}{2}\int_\Omega|\varphi|^2dx-\frac{1}{2\varepsilon}\int_\Omega(|\nabla g|^2+f^2)dx(\varepsilon\text{待定})\\
&\geq \frac{1-\varepsilon}{4}\int_\Omega |\nabla\varphi|^2dx+\left(\frac{1-\varepsilon}{4\mu}-\frac{\varepsilon}{2}\right)\int_\Omega|\varphi|^2dx-\frac{1}{2\varepsilon}\int_\Omega(|\nabla g|^2+f^2)dx\\
&\quad(\mu\text{为}Poincare\text{不等式中的常数})
\end{aligned}$$

取ε充分小,使得上式前两项的系数都大于0,而有

$$J(\varphi)\geq min\left\{\frac{1-\varepsilon}{4},\frac{1-\varepsilon}{4\mu}-\frac{\varepsilon}{2}\right\}||\varphi||^2_{H^1(\Omega}-\frac{1}{2\varepsilon}\int_\Omega(|\nabla g|^2+f^2)dx$$

从而有之.

由$a),b)$,我们可以证明J在$H_0^1(\Omega)$上能取到最小值.
因为J在$H^1(\Omega)$上有下界,由确界原理,有下确界

$$m=\inf_{\varphi\in H_0^1(\Omega)}J(\varphi)$$

其为聚点,

$$\exists\{w_n\}\subset H^1(\Omega),\ s.t.\ J(w_n)\to m$$

如果我们证明了$\{w_n\}$在$H_0^1(\Omega)$中有界, 那么由于$H_0^1(\Omega)$是自反空间(作为$H^1(\Omega)$的闭子空间,是一致凸的;或者直接用$Riesz$表示定理)及其$Eberlein$定理,有 $\{w_n\}$有一弱收敛的子列:

$$w_{n_k}\rightharpoonup w\in H_0^1(\Omega)$$

从而由J的弱下半连续性,有

$$m\leq J(w)\leq\liminf_{k\to\infty}J(w_{n_k})=\lim_{n\to\infty}J(w_n)=m$$

即有J在w处取得最小值. 往证$\{w_n\}$在$H_0^1(\Omega)$中有界.

$$\begin{aligned}
\frac{1}{2}\int_\Omega|\nabla w_n|^2dx &= J(w_n)-\int_\Omega\nabla g\cdot\nabla w_ndx+\int_\Omega fw_ndx\\
&\leq M+\frac{\varepsilon}{2}\int_\Omega|\nabla w_n|^2dx+\frac{1}{2\varepsilon}\int_\Omega|\nabla g|^2dx\\
&\quad+\frac{\varepsilon}{2}\int_\Omega|w_n|^2dx+\frac{1}{2\varepsilon}\int_\Omega f^2dx(M\text{是}J(w_n)\text{的一个上界})\\
&\leq M+\frac{\varepsilon(1+\mu)}{2}\int_\Omega|\nabla w_n|^2dx+\frac{1}{2\varepsilon}\int_\Omega(|\nabla g|^2+f^2)dx\\
&\quad(\mu\text{为}Poincare\text{不等式中的常数})
\end{aligned}$$

取ε充分小,可知$\int_\Omega|\nabla w_n|^2dx$有界,再由$Poincare$不等式,有$||w_n||_{H^1(\Omega)}$有界.得证.

4) $i)$ 存在性:

由3),可设J在$w \in H_0^1(\Omega)$上取得最小值,而由2),有$u = w + g$是问题$(D1)$的弱解.

$ii)$ 唯一性:

设u_1, u_2都是问题$(D1)$的弱解,令$u = u_1 - u_2$,则有

$$\int_\Omega \nabla u \cdot \nabla \psi dx = 0 \quad (\forall \psi \in H_0^1(\Omega))$$

取$\psi = u = (u_1 - g) - (u_2 - g) \in H_0^1(\Omega)$,代入有

$$\int_\Omega u^2 dx \le \mu \int_\Omega |\nabla u|^2 dx = 0$$

于是

$$u = 0,\ a.e.\text{于}\Omega$$

2 设Ω是$\mathbb{R}^n$中的有界区域,$\partial\Omega$充分光滑.考虑下列重调和$Poisson$方程的边值问题:

$$(D2)\begin{cases} \triangle^2 u(x) = f(x), x \in \Omega \\ u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0 \end{cases}$$

令$V = \{u \in C^\infty(\bar{\Omega}) \mid u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\}$,并令 $H_B^2(\Omega)$为V在$H^2(\Omega)$中的闭包.问题$(D2)$的弱解定义为满足下列条件的$u \in H_B^2(\Omega)$:

$$\int_\Omega \triangle u \cdot \triangle \varphi dx = \int_\Omega f\varphi dx, \forall \varphi \in V.$$

1) 证明:如果$u \in C^4(\Omega) \cap C^2(\bar{\Omega})$是问题$(D2)$的经典解,则$u$亦为它的弱解.

2) 令J为$H_B^2(\Omega)$上的下列泛函:

$$J(w) = \frac{1}{2}\int_\Omega |\triangle w|^2 dx - \int_\Omega fw dx,\ \forall w \in H_B^2(\Omega)$$

证明:$u \in H_B^2(\Omega)$是问题(D_2)的弱解当且仅当u是泛函J的最小元,即成立:$J(u) \le J(\varphi), \forall \varphi \in H_B^2(\Omega)$.

3) 应用变分原理证明:问题$(D2)$存在唯一的弱解.

证明:

1) 对任意的$u \in C^2(\Omega) \cap C^2(\bar{\Omega})$,由$Green$公式,有

$$\int_\Omega \triangle u \cdot \triangle \varphi dx = \int_\Omega \varphi \cdot \triangle^2 u dx + \int_{\partial\Omega} \left[\triangle u \frac{\partial \varphi}{\partial \nu} - \varphi \frac{\partial(\triangle u)}{\partial \nu}\right] ds = \int_\Omega f\varphi dx, \forall \varphi \in C^\infty(\bar{\Omega})$$

2) 明显的,若$u \in H^2(\Omega)$是问题$(D2)$的弱解,则有下列等式成立:

$$\int_\Omega \triangle u \cdot \triangle \varphi dx = \int_\Omega f\varphi dx, \forall \varphi \in H_B^2(\Omega)$$

a) 充分性: 设$u \in H_B^2(\Omega)$是J的最小元,对$\forall \varphi \in H_B^2(\Omega)$,作函数

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{R} \\ \varepsilon &\mapsto J(u + \varepsilon\varphi) \end{aligned}$$

有

$$\begin{aligned} F(\varepsilon) &= J(u + \varepsilon\varphi) \\ &= \frac{1}{2}\int_\Omega |\triangle(u + \varepsilon\varphi)|^2 dx - \int_\Omega f(u + \varepsilon\varphi) dx \\ &= \frac{1}{2}\int_\Omega |\triangle u|^2 dx + \varepsilon \int_\Omega \triangle u \cdot \triangle \varphi dx + \frac{1}{2}\varepsilon^2 \int_\Omega \varphi^2 dx - \int_\Omega fu dx - \varepsilon \int_\Omega f\varphi dx \end{aligned}$$

从而F可微且在$\varepsilon = 0$处取得最小值,由$Fermat$定理,

$$F'(0) = \int_\Omega \triangle u \cdot \triangle \varphi dx - \int_\Omega f\varphi dx = 0$$

即$u \in H_B^2(\Omega)$是问题$(D2)$的弱解.

b) 必要性:
设$u \in H_B^2(\Omega)$是问题$(D2)$的弱解,我们证明u是J的最小元.$\forall \varphi \in H_B^2(\Omega)$,将$\psi = \varphi - u \in H^2(\Omega)$作为试验函数代入弱解定义式有

$$\int_\Omega \triangle u \cdot \triangle \psi dx = \int_\Omega f \psi dx$$

于是

$$\begin{aligned} J(\varphi) &= J(\psi + u) \\ &= \frac{1}{2}\int_\Omega |\triangle(\psi+u)|^2 dx - \int_\Omega f(\psi+u)dx \\ &= \frac{1}{2}\int_\Omega |\triangle\psi|^2 dx + J(u) + \int_\Omega \triangle u \cdot \triangle\psi dx - \int_\Omega f\psi dx \\ &= \frac{1}{2}\int_\Omega |\triangle\psi|^2 dx + J(u) \\ &\geq J(u) \end{aligned}$$

3) 1) 存在性
由二阶椭圆型方程理论,设Ω是有界区域,$\partial\Omega \in C^2, f \in L^2(\Omega)$,问题

$$\begin{cases} -\triangle u = f, x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

有

$$||u||_{H^2(\Omega)} \leq C||f||_{L^2(\Omega)}$$

于是对$u \in H_B^2(\Omega)$,取$f = -\triangle u \in L^2(\Omega)$,有

$$||u||_{H^2(\Omega)} \leq C||\triangle u||_{L^2(\Omega)}$$

如此,便有

$$\begin{aligned} J(w) &= \frac{1}{2}\int_\Omega |\triangle w|^2 dx - \int_\Omega fw dx \\ &\geq C||w||^2_{H^2(\Omega)} - \frac{\varepsilon}{2}||u||^2_{L^2(\Omega)} - \frac{1}{2\varepsilon}||f||_{L^2(\Omega)} \\ &\geq C||w||^2_{H^2(\Omega)} - \frac{1}{2\varepsilon}||f||_{L^2(\Omega)} \end{aligned}$$

从而$J(w)$是强制的.再同上题,易证$J(w)$是弱下半连续的.于是$J(w)$能取到最小值$u \in H_B^2(\Omega)$,由2),即为问题$(D2)$的弱解.

2) 唯一性
若另有u'为问题$(D2)$的弱解,则

$$\int_\Omega \triangle(u-u') \cdot \triangle\varphi dx = 0, \forall\varphi \in C^\infty(\bar{\Omega})$$

即

$$\int_\Omega \triangle(u-u') \cdot \varphi dx = 0, \forall\varphi \in C^\infty(\bar{\Omega})$$

于是

$$\triangle(u-u') = 0, x \in \Omega$$

而

$$(u-u')|_{\partial\Omega} = u|_{\partial\Omega} - u'|_{\partial\Omega} = 0$$

由$Laplace$方程的极值原理,有$u = u'$,即证唯一性. □

注:3)的证明用到如下的定理:
设X是自反B^*空间(从而也是B空间),f是X上的实泛函,如果

1) f是弱下半连续的,即

$$X \supset \{x_n\} \rightharpoonup x \Rightarrow f(x) \leq \varliminf_{n\to\infty} f(x_n)$$

2) f是强制的,即

$$\lim_{||x||\to\infty} f(x) = +\infty$$

则f在X上能取到最小值.

证明:

1) f有下界
   若不然,

$$\exists \{x_n\} \subset X,\ s.t.\ f(x_n) < -n$$

   a) $\{x_n\}$有界,则由$Eberlein$定理,存在子列$\{x_{n_k}\} \rightharpoonup x$,而有

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = -\infty$$

   矛盾.

   b) $\{x_n\}$无界,而有子列$\{x_{n_k}\}$,使得$||x_{n_k}|| \to \infty(k \to \infty$,这又与$f$的强制性矛盾

2) f能达到下确界
   设$m = inf_{x\in X} f(x)$,则存在$\{x_n\} \subset X$,使得$f(x_n) \to m(n \to \infty)$,而$\{x_n\}$有界,有子列$\{x_{n_k}\} \rightharpoonup x$,于是

$$m \le f(x) \le \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = m$$

   即$f(x) = m$.证完.

3 设Ω是$\mathbb{R}^n$中的有界区域,$\partial\Omega$充分光滑,$a_{ij} \in L^\infty(\Omega)(i,j = 1,2,\cdots,n)$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $f^i \in L^2(\Omega)(i = 1,2,\cdots,n)$, $g \in H^1(\Omega)$.又设$a_{ij} = a_{ji}(i,j = 1,2,\cdots,n)$且有正常数$\lambda$使成立

$$a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega$$

考虑边值问题

$$(D3) \begin{cases} -D_j(a_{ij}D_iu) + cu = f + D_if^i, x \in \Omega \\ \gamma(u-g) = 0 \end{cases}$$

试用变分原理证明:问题$(D3)$存在唯一的弱解$u \in H^1(\Omega)$.
证明:

1) 弱解的定义:
   $u \in H^1(\Omega)$ 称为问题$(D3)$的弱解,如果$w = u - g \in H_0^1(\Omega)$满足

$$\int_\Omega a_{ij}D_i(w+g)\cdot D_j\varphi + c(w+g)\varphi dx = \int_\Omega f\varphi - f^iD_i\varphi dx, \forall \varphi \in H_0^1(\Omega)$$

2) 变分原理:
   设$w \in H_0^1(\Omega)$是泛函

$$\begin{aligned} J : H_0^1(\Omega) &\to \mathbb{R} \\ \psi &\mapsto \frac{1}{2}\int_\Omega a_{ij}D_i\psi\cdot D_j\psi + c\psi^2 dx \\ &\quad + \int_\Omega a_{ij}D_i\psi\cdot D_jg dx + \int_\Omega cg\psi dx - \int_\Omega (f\psi - f^iD_i\psi)dx \end{aligned}$$

   的最小元,则$u = w + g$是问题$(D3)$的弱解.
   $\forall \varphi \in H_0^1(\Omega)$,作函数

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{R} \\ \varepsilon &\mapsto J(w + \varepsilon\varphi) \end{aligned}$$

则

$$\begin{aligned}F(\varepsilon) &= J(w+\varepsilon\varphi)\\ &= \frac{1}{2}\int_\Omega a_{ij}D_i(w+\varepsilon\varphi)D_j(w+\varepsilon\varphi)+c(w+\varepsilon\varphi)^2dx\\ &\quad+\int_\Omega a_{ij}D_i(w+\varepsilon\varphi)D_jgdx+\int_\Omega cg(w+\varepsilon\varphi)dx\\ &\quad-\int_\Omega f(w+\varepsilon\varphi)-f^iD_i(w+\varepsilon\varphi)dx\end{aligned}$$

且$F(\varepsilon)$可微,且在$\varepsilon=0$处取得最小值,由$Fermat$定理,有

$$\begin{aligned}0 &= F'(\varepsilon)\\ &= \int_\Omega a_{ij}D_iwD_j\varphi+cw\varphi dx+\int_\Omega a_{ij}D_i\varphi D_jgdx\\ &\quad+\int_\Omega cg\varphi dx-\int_\Omega f\varphi-f^iD_igdx\end{aligned}$$

从而$u=w+g$是问题$(D3)$的弱解.

3) J是弱下半连续的

令$J=J_1+J_2$,其中

$$J_1(\varphi)=\frac{1}{2}\int_\Omega a_{ij}D_i\psi\cdot D_j\psi+c\psi^2dx$$

$$J_2(\psi)=\int_\Omega a_{ij}D_i\psi\cdot D_jgdx+\int_\Omega cg\psi dx-\int_\Omega(f\psi-f^iD_i\psi)dx$$

对于$\forall\psi_m\rightharpoonup\psi$(于$H_0^1(\Omega)$),

对J_2,其中为$H_0^1(\Omega)$上的线性连续泛函,故有$J_2(\psi_m)\to J_2(\psi)(m\to\infty)$;

对J_1,令$\varphi_m=\psi_m-\psi$,则$\varphi_m\rightharpoonup 0$(于$H_0^1(\Omega)$),有

$$\begin{aligned}J_1(\psi_m) &= J_1(\varphi_m+\psi)\\ &= \frac{1}{2}\int_\Omega a_{ij}D_i(\varphi_m+\psi)D_j(\varphi_m+\psi)+c(\varphi_m+\psi)^2dx\\ &\geq \int_\Omega a_{ij}D_i\varphi D_j\psi dx+\int_\Omega c\varphi_m\psi dx+J_1(\psi)\end{aligned}$$

两边令$m\to\infty$,取下极限有

$$\varliminf_{m\to\infty}J_1(\psi_m)\geq J_1(\psi)$$

从而J是弱下半连续的.

4) 存在常数$c_0\leq 0$,使得当$c\geq c_0$时,J是强制的

$$\begin{aligned}J_(\psi) &= \frac{1}{2}\int_\Omega a_{ij}D_i\psi\cdot D_j\psi+c\psi^2dx\\ &\quad+\int_\Omega a_{ij}D_i\psi\cdot D_jgdx+\int_\Omega cg\psi dx-\int_\Omega(f\psi-f^iD_i\psi)dx\\ &\geq \frac{1}{2}\int_\Omega a_{ij}D_i\psi D_j\psi dx+\frac{c_0}{2}\int_\Omega\psi^2dx-\frac{\varepsilon}{2}\int_\Omega a_{ij}D_i\psi D_j\psi dx\\ &\quad-\frac{1}{2\varepsilon}\int_\Omega a_{ij}D_igD_jgdx+\frac{c_0\varepsilon}{2}\int_\Omega\psi^2dx+\frac{c_0}{2\varepsilon}\int_\Omega|g|^2dx\\ &\quad-\frac{\varepsilon}{2}\int_\Omega\psi^2dx-\frac{1}{2\varepsilon}\int_\Omega|f|^2dx-\frac{\varepsilon}{2}\int_\Omega|\nabla\psi|^2dx-\frac{1}{2\varepsilon}\int_\Omega f^if^idx\\ &= \left(\frac{\lambda}{2}-\frac{\lambda\varepsilon}{2}-\frac{\varepsilon}{2}\right)\int_\Omega|\nabla\psi|^2dx+\left(\frac{c_0}{2}+\frac{c_0\varepsilon}{2}-\frac{\varepsilon}{2}\right)\int_\Omega\psi^2dx\\ &\quad-\frac{1}{2\varepsilon}\int_\Omega a_{ij}D_igD_jgdx+\frac{c_0}{2\varepsilon}\int_\Omega|g|^2dx-\frac{1}{2\varepsilon}\int_\Omega f^2dx-\frac{1}{2\varepsilon}\int_\Omega f^if^idx\\ &\geq min\left\{\frac{\lambda-\lambda\varepsilon-\varepsilon}{4},\frac{\lambda-\lambda\varepsilon-\varepsilon}{4\mu}+\frac{c_0}{2}+\frac{c_0\varepsilon}{2}-\frac{\varepsilon}{2}\right\}||\psi||_{H^1(\Omega)}\\ &\quad-\frac{1}{2\varepsilon}\int_\Omega a_{ij}D_igD_jgdx+\frac{c_0}{2\varepsilon}\int_\Omega|g|^2dx-\frac{1}{2\varepsilon}\int_\Omega f^2dx-\frac{1}{2\varepsilon}\int_\Omega f^if^idx\end{aligned}$$

现如今取$\varepsilon > 0$充分小,使得

$$\frac{\lambda - \lambda\varepsilon - \varepsilon}{4} > 0, \quad \frac{\lambda - \lambda\varepsilon - \varepsilon}{4\mu} - \frac{\varepsilon}{2} > 0$$

取

$$c_0 \in \left(\max\left\{ -\frac{2}{1+\varepsilon}\left[\frac{\lambda - \lambda\varepsilon - \varepsilon}{4\mu} - \frac{\varepsilon}{2}\right], -\frac{\lambda}{\mu} \right\}, 0 \right)$$

则有J强制.

5) 弱解的存在性

由3), 4)知泛函J能于$w \in H_0^1(\Omega)$出取得最小值,再由2)知$u = w + g$即是问题$(D3)$的弱解.

6) 弱解的唯一性

设$u_1, u_2 \in H^1(\Omega)$均为问题$(D3)$的弱解,令

$$w_1 = u_1 - g, w_2 = u_2 - g, w = w_1 - w_2 \in H_0^1(\Omega)$$

则由弱解的定义式有

$$\int_\Omega a_{ij} D_i w D_j \varphi + cw\varphi dx = 0, \forall \varphi \in H_0^1(\Omega)$$

特别取$\varphi = w$,有

$$\begin{aligned} 0 &= \int_\Omega a_{ij} D_i w D_j w + cw^2 dx \\ &\geq \lambda \int_\Omega |\nabla w|^2 dx + c_0 \int_\Omega w^2 dx \\ &\geq \left(\frac{\lambda}{\mu} + c_0\right) \int_\Omega |w|^2 dx \end{aligned}$$

从而

$$\int_\Omega |w|^2 dx = 0, w = 0, a.e.于\Omega, 即有w_1 = w_2, 而u_1 = u_2, a.e.于\Omega$$

4 设Ω是$\mathbb{R}^n$中的区域,$a_{ij} \in W^{1,\infty}(\Omega)(i,j = 1,2,\cdots,n), c \in L^\infty(\Omega), f \in L^2(\Omega)$,且$a_{ij}$满足第三题的正定性条件,设$u \in H^1(\Omega)$下列方程在区域$\Omega$上的弱解:

$$-D_j(a_{ij} D_i u) + cu = f$$

证明:$u \in H_{loc}^2(\Omega)$.

证明:$\forall \Omega' \subset\subset \Omega$,作$\Omega$相对于$\Omega'$的截断函数$\eta$,满足

$$0 \leq \eta(x) \leq 1于\Omega; \quad \eta(x) = 1于\Omega'; \quad |\nabla \eta| \leq C, 于\Omega$$

其中$C = C(n, \Omega, dist(\Omega's, \partial\Omega))$.

令$\varphi(x) = \triangle_h^{k*}(\eta^2 \triangle_h^k u)(k = 1,2,\cdots,n)$,则当$h$充分小时,$\varphi(x) \in H_0^1(\Omega)$,作为试验函数代入问题$(D4)$的弱解定义式中有

$$\int_\Omega a_{ij} D_i u D_j \varphi dx + \int_\Omega cu\varphi dx = \int_\Omega f\varphi dx$$

$$\begin{aligned} 左边 &= \int_\Omega a_{ij} D_i u D_j \varphi dx + \int_\Omega cu\varphi dx \\ &= \int_\Omega a_{ij} D_i u D_j(\triangle_h^{k*}(\eta^2 \triangle_h^k u))dx + \int_\Omega cu \triangle_h^{k*}(\eta^2 \triangle_h^k u) dx \\ &= \int_\Omega \triangle_h^k (a_{ij} D_i u) D_j(\eta^2 \triangle_h^k u))dx - C\int_\Omega u \triangle_h^{k*}(\eta^2 \triangle_h^k u) dx \\ &\geq \int_\Omega (T_h^k a_{ij} D_i \triangle_h^k u + \triangle_h^k a_{ij} D_i u) D_j(\eta^2 \triangle_h^k u))dx - C\varepsilon \int_\Omega |\triangle_h^{k*}(\eta^2 \triangle_h^k u)|^2 dx - \frac{C}{\varepsilon}\int_\Omega u^2 dx \\ &\geq \int_\Omega (T_h^k a_{ij} D_i \triangle_h^k u + \triangle_h^k a_{ij} D_i u) D_j(\eta^2 \triangle_h^k u) dx - C\varepsilon \int_\Omega |\triangle_h^{k*}(\eta^2 \triangle_h^k u)|^2 dx - \frac{C}{\varepsilon}\int_\Omega u^2 dx \\ &= \int_\Omega \eta^2 T_h^k a_{ij} D_i \triangle_h^k u D_j \triangle_h^k u dx + 2\int_\Omega T_h^k a_{ij}(\eta D_i \triangle_h^k u)(\triangle_h^k u D_j \eta) dx \\ &\quad -C\int_\Omega D_i u D_j(\eta^2 \triangle_h^k u) dx - C\varepsilon \int_\Omega |\triangle_h^{k*}(\eta^2 \triangle_h^k u)|^2 dx - \frac{C}{\varepsilon}\int_\Omega u^2 dx \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

其中

$$\begin{aligned}
I_1 &= \int_\Omega \eta^2 T_h^k a_{ij} D_i \triangle_h^k u D_j \triangle_h^k u dx + 2\int_\Omega T_h^k a_{ij}(\eta D_i \triangle_h^k u)(\triangle_h^k u D_j \eta)dx \\
&\geq \int_\Omega \eta^2 T_h^k a_{ij} D_i \triangle_h^k u D_j \triangle_h^k u dx - \varepsilon \int_\Omega \eta^2 T_h^k a_{ij} D_i \triangle_h^k u D_j \triangle_h^k u dx - \frac{1}{\varepsilon}\int_\Omega T_h^k a_{ij} |\triangle_h^k u|^2 D_i \eta D_j \eta dx \\
&\geq (1-\varepsilon)\int_{\Omega'} T_h^k a_{ij} D_i \triangle_h^k u D_j \triangle_h^k u dx - \frac{1}{\varepsilon}\int_\Omega T_h^k a_{ij} |\triangle_h^k u|^2 D_i \eta D_j \eta dx \\
&\geq \lambda(1-\varepsilon)\int_{\Omega'} |\triangle_h^k \nabla u|^2 dx - \frac{C}{\varepsilon}\int_\Omega |\triangle_h^k u|^2 dx \\
&\geq \lambda(1-\varepsilon)\int_{\Omega'} |\triangle_h^k \nabla u|^2 dx - \frac{C}{\varepsilon}\int_\Omega |\nabla u|^2 dx
\end{aligned}$$

$$\begin{aligned}
I_2 &= -C\int_\Omega D_i u D_j(\eta^2 \triangle_h^k u)dx \\
&\geq -\frac{C}{\varepsilon}\int_\Omega |\nabla u|^2 dx - C\varepsilon\int_\Omega |\nabla(\eta^2 \triangle_h^k u)|^2 dx \\
&= -\frac{C}{\varepsilon}\int_\Omega |\nabla u|^2 dx - 2C\int_\Omega |2\eta \triangle_h^k u \nabla \eta|^2 dx - 2C\varepsilon\int_\Omega \eta^2 |\triangle_h^k \nabla u|^2 dx \\
&\geq -\frac{C}{\varepsilon}\int_\Omega |\nabla u|^2 dx - 8C\varepsilon\int_\Omega |\nabla u|^2 dx - 2C\varepsilon\int_\Omega |\triangle_h^k \nabla u|^2 dx
\end{aligned}$$

$$I_3 = -\frac{C}{\varepsilon}\int_\Omega u^2 dx$$

$$\begin{aligned}
I_4 &= -C\varepsilon\int_\Omega |\triangle_h^{k*}(\eta^2 \triangle_h^k u)|^2 dx \\
&\geq -2C\varepsilon\int_\Omega |2\eta \triangle_h^k u \nabla \eta|^2 dx - 2C\varepsilon\int_\Omega \eta^2 |\triangle_h^k \nabla u|^2 dx \\
&\geq -8C\varepsilon\int_\Omega |\nabla u|^2 dx - 2C\varepsilon\int_{\Omega'} |\triangle_h^k \nabla u|^2 dx
\end{aligned}$$

$$\begin{aligned}
\text{右边} &= \int_\Omega f\varphi dx \\
&\leq \frac{1}{2\varepsilon}\int_\Omega f^2 dx + \frac{\varepsilon}{2}\int_\Omega \varphi^2 dx \\
&\leq \frac{1}{2\varepsilon}\int_\Omega f^2 dx + \frac{\varepsilon}{2}\left[8\int_\Omega |\triangle u|^2 dx + 2\int_\Omega |\triangle_k^h \nabla u|^2 dx\right] \\
&= \frac{1}{2\varepsilon}\int_\Omega f^2 dx + 4\varepsilon\int_\Omega |\triangle u|^2 dx + \varepsilon\int_{\Omega'} |\triangle_k^h \nabla u|^2 dx
\end{aligned}$$

由上即得

$$\lambda(1-2\varepsilon)\int_{\Omega'} |\triangle_h^k \nabla u|^2 dx \leq C(||u||^2_{H^1(\Omega} + ||f||^2_{L^2(\Omega)})$$

从而

$$\int_{\Omega'} |D_k \nabla u|^2 dx \leq C(||u||^2_{H^1(\Omega)} + ||f||^2_{L^2(\Omega)})$$

$$||u||_{H^2(\Omega')} \leq C(||u||_{H^1(\Omega)} + ||f||_{L^2(\Omega)})$$

$$u \in H^2_{loc}(\Omega) \ \square$$

5 设$f \in L^2(\mathbb{R}^n_+)$,而$u \in H^1(\mathbb{R}^n_+)$是下列方程在区域$\mathbb{R}^n_+$上的弱解:

$$(D4)\begin{cases} -\triangle u = f(x), x \in \mathbb{R}^n_+ \\ u|_{\partial \mathbb{R}^n_+} = 0 \end{cases}$$

这里$\mathbb{R}_+^n=\{x=(x',x_n)\in\mathbb{R}_+^n \mid x'\in\mathbb{R}^{n-1},x_n>0\}$.
证明:$u\in H^2(\mathbb{R}_+^n)$.
证明:取$\mathbb{R}_+^n$的一致局部有限开覆盖$\{\Omega_i\}$,即

$$\exists N>0,\ s.t.K\text{紧}\Rightarrow Card\{i\mid \Omega_i\cap K\neq\emptyset\}\leq N$$

再作从属于$\{\Omega_i\}$的单位分解$\{\eta_i\}$,使得

$$supp\eta_i\subset\Omega_i\text{且}\eta_i,\nabla\eta_i,\nabla D_j\eta_i\text{一致的有界}$$

则由$Poisson$方程$Dirichlet$问题弱解的内部正则性与近边正则性有

$$||u||^2_{H^2(\mathbb{R}_+^n)}\leq C\sum_{i=1}^{\infty}||u||^2_{H^2(\Omega_i\cap\mathbb{R}_+^n)}\leq C\sum_{i=1}^{\infty}\left[||u||^2_{H^1(\Omega_i\cap\mathbb{R}_+^n)}+||f||^2_{L^2(\Omega_i\cap\mathbb{R}_+^n)}\right]\leq C\left[||u||^2_{H^1(\mathbb{R}_+^n)}+||f||^2_{L^2(\mathbb{R}_+^n)}\right]$$

即有

$$u\in H^2(\mathbb{R}_+^n)$$

另证如下:

1) $H_0^1(\mathbb{R}_+^n)=\left\{u\in H^1(\mathbb{R}_+^n)\mid \gamma u=0\right\}$.事实上,显然,$H_0^1(\mathbb{R}_+^n)\subset\left\{u\in H^1(\mathbb{R}_+^n)\mid\gamma u=0\right\}$.下证反过来的包含关系,为此设$u\in H^1(\mathbb{R}_+^n))$,且$\gamma u=0$.则对$\forall\varphi\in C_0^\infty(\mathbb{R}^n))$有$\varphi u\in H_0^1(\mathbb{R}_+^n))$,对每个$R>0$,取$\varphi_R\in C_0^\infty(\mathbb{R}^n)$使得$\varphi_R(x)=1$于$\{x\mid |x|\leq R\}$;$\varphi_R(x)=0$于$\{x\mid |x|\geq R+1\}$; $|\nabla\varphi|\leq C$(C与R无关).令$u_R=\varphi_R u$,则

$$u_R\in H_0^1(\mathbb{R}_+^n)\text{且}\lim_{R\to\infty}||u_R-u||_{H^1(\mathbb{R}_+^n)}=0$$

所以$u\in H_0^1(\mathbb{R}_+^n)$.

2) $u\in H^2(\Omega)$.由$u\in H^1(\mathbb{R}_+^n)$及$u|_{\partial\mathbb{R}_+^n}=0$知$u\in H_0^1(\mathbb{R}_+^n)$.$\forall 1\leq k\leq n-1$及$h>0$,有$\varphi=\triangle_h^{k*}\triangle_h^k u\in H_0^1(\mathbb{R}_+^n)$,作为试验函数代入问题($D5$)弱解的定义式,有

$$\begin{aligned}\int_{\mathbb{R}_+^n}|\nabla D_k u|^2dx &\leq \int_{\mathbb{R}_+^n}|\triangle_h^k\nabla u|^2dx=\int_{\mathbb{R}_+^n}\triangle_h^k\nabla u\cdot\triangle_h^k\nabla u dx=\int_{\mathbb{R}_+^n}\nabla u\cdot\nabla\triangle_h^{k*}\triangle_h^k u dx\\ &= \int_{\mathbb{R}_+^n}\nabla u\cdot\nabla\varphi dx=\int_{\mathbb{R}_+^n}f\varphi dx\\ &\leq \frac{1}{2}\int_{\mathbb{R}_+^n}f^2dx+\frac{1}{2}\int_{\mathbb{R}_+^n}|\triangle_h^{k*}\triangle_h^k u|^2dx\leq\frac{1}{2}\int_{\mathbb{R}_+^n}f^2dx+\frac{1}{2}\int_{\mathbb{R}_+^n}|\triangle_h^k\nabla u|^2dx\\ &\leq \frac{1}{2}\int_{\mathbb{R}_+^n}f^2dx+\frac{1}{2}|\nabla D_k u|^2dx\end{aligned}$$

于是有

$$\int_{\mathbb{R}_+^n}|\nabla D_k u|^2dx\leq\int_{\mathbb{R}_+^n}f^2dx,k=1,2,\cdots,n$$

即

$$D_{ij}u\in L^2(\mathbb{R}_+^n),1\leq i\leq n-1,1\leq j\leq n$$

而

$$D_{nn}u=-\sum_{k=1}^{n-1}D_{ii}u-f\in L^2(\mathbb{R}_+^n)$$

故有

$$u\in H^2(\mathbb{R}_+^n)\ \square$$