

Rich-club connectivity dominates assortativity and transitivity of complex networks

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Rich-club, assortativity and clustering coefficients are frequently-used measures to estimate topological properties of complex networks. Here we find that the connectivity among a very small portion of the richest nodes can dominate the assortativity and clustering coefficients of a large network, which reveals that the rich-club connectivity is leveraged throughout the network. Our study suggests that more attention should be paid to the organization pattern of rich nodes, for the structure of a complex system as a whole is determined by the associations between the most influential individuals. Moreover, by manipulating the connectivity pattern in a very small rich-club, it is sufficient to produce a network with desired assortativity or transitivity. Conversely, our findings offer a simple explanation for the observed assortativity and transitivity in many real world networks — such biases can be explained by the connectivities among the richest nodes.

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After ten years of explosive growth, fruitful measures based on statistical physics have been proposed for analyzing all kinds of complex networks [1]. Measures such as degree distribution, average degree, clustering coefficient, assortativity coefficient, and average shortest-path length, are now widely used in almost all complex networks to estimate their topological properties. For example, clustering coefficient [2] is used to measure the transitivity property of a network. If a social network has a high clustering coefficient, it means that the friends of someone are also likely to be friends themselves [3].

A second popular measure is the assortativity coefficient which defines the mixing pattern among the nodes. A positive coefficient indicates that nodes with similar degrees tend to be connected to each other (assortative mixing), while a negative coefficient captures the opposite case in which very different degree nodes are connected (disassortative mixing) [3, 4]. Although the above calculations on assortativity and transitivity may be useful in many situations, the actual validity of these measures to capture the true assortativity and transitivity of the network has not been verified. In particular, the effectiveness of assortativity coefficient in some specific networks has been critically examined recently [5, 6].

Many real networks display a skewed degree distribution [7], so a small number of nodes possess much higher degrees than the overwhelming majority. Nonetheless, it is necessary to be cautious in applying such statistical measures as the actual value of most statistics (e.g., assortativity and clustering coefficients) is the statistical average of a whole network, and this averaging process may conceal the prominent effect of the richest elements [8]. Furthermore, it is already clear that the small num-

ber of rich nodes play a central role in static and dynamic processes on complex networks, such as targeted attack [9], cascade failure [10], and disease spreading [11]. Therefore, more attention should be paid to rich nodes when analyzing finite-size network data [5]. In particular, it is interesting to analyze the organization pattern of rich nodes [12], such as whether rich nodes tend to connect to one another, or with the rest of nodes [13].

Compared with a corresponding randomized network, if rich nodes are interconnected to one another more intensely than to low-degree nodes, the network is said to have a rich-club property [14–18]. Note that, rich-club only describes the property of rich nodes, and it is not a statistical average over the entire network. Rich-club is therefore different from the statistics that are based on the averaged results over all nodes (like clustering and assortativity coefficients). In this study, we demonstrate that the connections among a very small portion (no more than 0.5%) of rich nodes control the statistical properties of the entire complex networks, especially assortativity and transitivity properties. We find that adding a small number of extra links among rich nodes can significantly increase an assortativity coefficient to be positive, and raise a low clustering coefficient to a high value. These results show that it is possible to engineer the transitive or assortative features of a large complex network just by altering the wiring structure within a very small rich-club. Finally, this work allows us to explain the observed assortativity/transitivity of various real world networks (e.g. the Internet) by studying the connectivity between the richest nodes. That is, the structure of a complex system is mostly determined by the associations between the most influential individuals.

We select the top 0.5% of the highest degree nodes as rich nodes in a network and manipulate the connections among them. First we make rich nodes fully connected

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TABLE I: Statistics of nine undirected networks: number of nodes n , average degree $\langle k \rangle$, the exponent of degree distribution if the distribution follows a power law: α (or “-” if not), structural cutoff degree $k_s = \sqrt{\langle k \rangle n}$ [19], maximal degree k_{max} , assortativity coefficient r [4], clustering coefficient c [2], and average shortest-path length l . SW is the network generated by the small-world model [2], ER is the network generated by Erdős-Rényi model [20], PG is the network of US power grid [7], COND is the network of scientists who work on condensed matter [21], BA is the network generated by the scale-free model [7], EPA is the network from the pages linking to www.epa.gov [22], PFP is the network generated by the model for the Internet topology [23], AS is the network of the Internet topology at the level of autonomous systems [24] and BOOK is the word adjacency network of text from Darwin’s “The Origin of Species” [25]. The proportion of rich nodes in all the networks is 0.5% except the network of COND. We select less proportion (0.2%) nodes as rich nodes in COND, because it has larger scale (more nodes) than other networks. For r , c and l , the first row is the value when rich nodes do not connect to other rich nodes (without rich-club), and the second row is the value when rich nodes completely connect to each other (with rich-club).

Network	SW	ER	PG	COND	BA	EPA	PFP	AS	BOOK
n	5000	5000	4941	16726	5000	4772	5000	5375	7724
$\langle k \rangle$	6.0	10.0	2.7	5.7	6.0	3.7	6.0	3.9	11.4
α	—	—	—	—	3.0	2.0	2.2	2.2	1.9
k_{max}	15.5 ± 3.5	23.4 ± 1.6	19	107	218.6 ± 43.4	175	1258.8 ± 349.0	1193	2568
k_s	173.2	223.6	115.4	308.8	173.2	132.9	173.2	144.8	296.7
k_{max}/k_s	0.09	0.10	0.16	0.35	1.26	1.32	7.26	8.24	8.66
r	0.00 ± 0.00	0.00 ± 0.00	-0.01	0.17	-0.08 ± 0.01	-0.31	-0.25 ± 0.04	-0.19	-0.24
	0.69 ± 0.00	0.39 ± 0.00	0.60	0.32	0.04 ± 0.02	-0.15	-0.24 ± 0.04	-0.19	-0.24
c	0.44 ± 0.00	0.00 ± 0.00	0.08	0.62	0.00 ± 0.00	0.04	0.15 ± 0.01	0.10	0.21
	0.44 ± 0.00	0.00 ± 0.00	0.08	0.62	0.02 ± 0.00	0.07	0.28 ± 0.02	0.26	0.41
l	7.85 ± 0.05	3.94 ± 0.01	6.63	6.64	4.11 ± 0.02	4.63	3.17 ± 0.06	3.95	2.87
	7.33 ± 0.03	3.94 ± 0.01	6.37	6.37	3.94 ± 0.02	3.97	3.04 ± 0.05	3.60	2.77

to one another, so they form a completely connected rich-club. Secondly, we completely eradicate the edges among these rich nodes, so that the network has no rich-club. The topological structure is the same for the above two networks except for the connection pattern among rich nodes. Then we calculate the frequently-used statistics for the above two networks respectively to compare how the absence and presence of a rich-club affects the statistical properties of the whole network.

Table I lists the results of nine undirected networks (including five real networks and four model networks) arranged with k_{max}/k_s increasing. The value of the structural cutoff degree k_s can be regarded as the first approximation in a scale-free network [19]. Here k_{max}/k_s is a convenient index that can be used in complex networks with any degree distribution to show the proportion of links (or degrees) the rich nodes possess in comparison with the rest nodes in a network. Lower k_{max}/k_s means that the degrees of rich nodes are close to the majority of nodes, while a high k_{max}/k_s indicates that the degrees of rich nodes are far larger than the rest.

The results in Table I show whether a very small proportion of rich nodes forms a club can partly control the two important statistics: assortativity coefficient r and clustering coefficient c . Based on the different values of k_{max}/k_s , complex networks fall into two distinct groups. In the networks with low k_{max}/k_s like SW, ER, PG, COND, BA and PG, the values of r are largely determined by the rich-club. But for the networks with high k_{max}/k_s such as PFP, AS and BOOK, the values of c are largely determined by the rich-club.

Now we analyze how the rich-club connectivity domi-

nates r . Recently, the effectiveness of r in some specific networks has been queried. In our previous work [5], we found that superrich nodes (degree much larger than the natural cutoff value [19]) can strongly influence r . Meanwhile, another work showed that the highly heterogeneous (scale-free) network with “natural” degree mixing has a disassortative coefficient [6]. These studies indicate that r is always strongly negative for some specific networks [16]. In Table I, we also find that r is strongly negative for the networks with a high k_{max}/k_s (i.e., with superrich nodes [5]), such as PFP, AS and BOOK.

While the above studies focus on the effect of rich nodes, in this work we pay more attention to how the *organization* of rich nodes (to form a rich-club or not) affects r . For networks with low k_{max}/k_s and the absence of a rich-club such as SW, ER and PG, the values of r are near zero, which indicates that these networks are neutral mixing. But the counterparts with the presence of a rich-club show a surprisingly positive r , which implies that these networks have assortative mixing properties. It is obvious that the mixing patterns of more than 99.5% nodes remain unchanged, so this metamorphosis is induced by the absence and presence of the rich-club. For the networks COND, BA and EPA, our results again imply that the connections among no more than 0.5% rich nodes can make r become much more positive.

For networks with a high k_{max}/k_s , such as PFP, AS and BOOK, the presence of a rich-club does slightly affect r , while it strongly affects c . Traditionally, high c indicates that the friends of someone are also likely to be friends themselves. A highly assortative network often implies a high c as nodes with similar degrees will con-

nect to each other [26] and form multiscale communities [3]. But in a highly disassortative network, a high-degree node tends to connect to a low-degree node, which in turn connects to another high-degree node, and this high-low-high-low connection circle will lead to a low c . It is therefore not obvious why a high c emerges in disassortative networks like PFP, AS and BOOK.

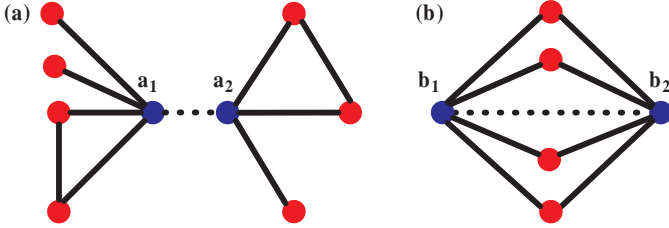


FIG. 1: (Color online) (a) Whether rich nodes a_1 and a_2 are to be connected will not significantly affect clustering coefficient c , while (b) whether rich nodes b_1 and b_2 form a rich-club strongly affects c .

Although the high values of c in the high disassortative networks with rich-club are contrary to our intuition, this phenomenon can be partly explained by considering the effect of the rich-club in more detail. As has been shown in Fig. 1(a), if rich nodes a_1 and a_2 are connected to each other, the value of c for this network will only change slightly. While if rich nodes b_1 and b_2 are connected to each other as is shown in Fig. 1(b), the network will show a high c . Moreover, the scenario in Fig. 1(b) shows that a high c does not always imply that the friends of someone are also likely to be connected for some specific networks. For example, even if b_1 connecting to b_2 makes the network in Fig. 1(b) show a high c , the other four low-degree nodes do not connect to each other either.

For other statistics such as average degree, degree distribution, and average shortest-path length, it is easy to guess how the presence or absence of a rich-club can influence them. Because the proportion of rich nodes manipulated here is no more than 0.5%, the degree distribution and average degree remain largely unchanged whether a network has a rich-club or not. Another statistic that is vulnerable to rich-club phenomena is average shortest-path length l [13]. Rich nodes often act as a traffic hub and provide a large selection of shortcuts, hence we can guess that a network without rich-club may lose the efficiency compared with its rich-club counterpart. For all the nine networks in Table I, this conjecture is right, for the presence and absence of a rich-club also strongly affects l , although not as strong as r and c .

It should be noted that a large k_{max}/k_s can reduce l more significantly than the presence of a rich-club. For networks with the same average degree, such as SW and PFP in Table I, the degree of the richest node in SW is far lower than that in PFP, so the value of l in the former is larger than the latter. In the network with low k_{max}/k_s (SW), every rich node only connects to a small number of nodes and they can only provide sparse shortcuts for

other nodes, so the network has a longer l [7.33 \sim 7.85]. In the network with high k_{max}/k_s (PFP), rich nodes have to connect to a huge number of low-degree nodes, so rich nodes provide a lot of shortcuts to low-degree nodes and the network has a shorter l [3.04 \sim 3.17].

Whether a network should be considered as having a rich-club has been discussed directly in some specific networks. For example, whether the network of Internet has a rich-club has been debated [13, 14, 16], and there is still not a clear conclusion. Furthermore, a dilemma of rich-club definition occurred in [18] and is shown in Fig. 2. In the definition of Zhou and Mondragón [13], they only study whether rich nodes are more likely to interconnect than to low-degree nodes, so that our toy model is therefore regarded as having a rich-club. However, Colizza *et al.* believe that rich-club should be inferred by a comparison of the original network with its randomized counterparts (reference network) [27] to avoid the false inference of rich-club in non-rich-club networks. Consequently, for the toy model in Fig. 2, the method in [14] will run into a dilemma, for the original network and its randomized version show the same structure.

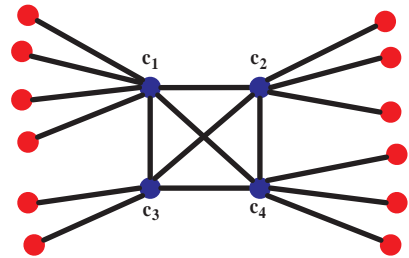


FIG. 2: (Color online) A toy model to show the dilemma of rich-club definition [18]. Rich nodes c_1 - c_4 have larger degrees and form a subnetwork in which rich nodes are completely connected to one another, so the network has a rich-club according to the definition in [13, 16]. But there is no rich-club using the definition in [14], for c_1 - c_4 are always connected to each other too in its corresponding randomized network.

To harmonize this contradiction, the frequently-used statistics can be used to judge whether a network has a rich-club. For the network with low k_{max}/k_s , we prefer to use c as the primary statistic; while for the network with high k_{max}/k_s , we can use r instead. Our framework is based on whether the statistics of the original network are strongly affected by the absence and presence of a rich-club. If the statistics of the original network are more similar to its fully-connected rich-club counterparts, and are far away to its non-rich-club counterparts, we can conclude that the network has a rich-club. Conversely, if this is not the case then we would conclude that the network has no rich-club.

We now use this new method to judge whether the Internet has a rich-club. We list the statistics r , c , and l for the four versions of the Internet network in Table II: the network without rich-club, the original network, the network with rich-club and the corresponding ran-

domized network. The properties of the original network are found to be more close to the network with rich-club, and are substantially different to the network without rich-club. This is especially obvious for the value of c , so it is easy to conclude that the network has a rich-club.

TABLE II: Statistics on four versions of the Internet network at the level of autonomous systems [24]: the number of total links among rich nodes m , clustering coefficient c [2], assortativity coefficient r [4], and average shortest-path length l . We choose 27 nodes (0.5% of the whole nodes) with the highest degrees as rich nodes. Origin stands for the original network; non-rich-club stands for the original network deleted the links among rich nodes; rich-club stands for the original network in which rich nodes are completely connected to each other; random stands for the randomized version of the original network generated by the random mixing method [27].

Network	non-rich-club	origin	rich-club	random
m	0	148	351	209.4 ± 10.4
c	0.10	0.24	0.26	0.13 ± 0.00
r	-0.19	-0.18	-0.19	-0.18 ± 0.00
l	3.95	3.70	3.60	3.54 ± 0.01

Our new method for measuring rich-club can provide a more satisfactory and impartial judgement on whether a network has a rich-club. The new method does not depend explicitly on how many links there are among rich nodes as previous measures that have been taken [14]. Rather, our approach is to directly measure the effect that the rich-club has on the properties of the whole network. Nonetheless, we are not suggesting that the existing tools for detecting rich-clubs should be abandoned. The controversy over whether particular networks have a rich-club is due to the tension between what are meant with evocative names and description (as are associated with the term “rich-club”) and what is actually being measured with various statistics. A more appropriate question is what effect these measured properties have on the network structure and dynamics.

In this work, we focus on how the rich-club affects the basic statistics of complex networks, especially assortativity and clustering coefficients. Our findings uncover the effect of the organization of rich nodes, which

leads to a better understanding of the behavior of a complex system. These results show that just by altering the wiring structure within a very small rich-club one can engineer the transitive or assortative features of a large complex network. The organization of rich nodes is crucial because it can strongly affect our understanding for the whole topological properties of the network. Our study indicates that in complex systems the social cohesion (that is the assortativity or transitivity) of a large community is determined by connectivity among the leaders (the rich-club). This study also confirms that although some measures developed in the framework of statistical physics provide a powerful tool for analyzing the organization of complex network, in specific situations they are very sensitive to a small local structure (the connectivity among a very small rich-club).

Nonetheless, the networks in Table I are not carefully selected on purpose, and our findings do provide a simple explanation for the observed properties of many real world networks. When examining such networks, we need not ask why they exhibit assortativity or transitivity, but rather how the rich nodes are connected and why they are connected in this way. For example, in the case of the Internet the rich nodes form a very strong rich-club (the various routers are interconnected) and it is this property that determines the transitivity of the entire network.

Conversely, in some situations (such as to control epidemic spread or information flow) it is useful to manipulate the assortativity and transitivity of a large network. Our results provide a cheap and easy way to do this: just manipulate the connections among the rich-club members. Followed the work in [8], an interesting question to be pursued in future would then be the investigation of how rich-club affects these important dynamic processes in weighted and/or directed networks.

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