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A FEW FACTS ABOUT HARMONIC MAPS

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ABSTRACT. We state some basic facts about the harmonic maps. This is [1, 1.5].

Proposition 1. *Let*

1. $\Phi : M \to M$ *a* C^2*-diffeomorphism;*
2. *and* $u \in C^2(M; N)$ *is a harmonic map with respect to* (M, g).

Then

$$u \circ \Phi \in C^2(M; N) \text{ is a harmonic map with respect to } (M, \Phi^* g).$$

Proof. For $v \in C^2(M; N)$, we have

$$\frac{1}{2} \int_M |\nabla v|_g^2 \, \mathrm{d}v_g = \frac{1}{2} \int_M |\nabla (v \circ \Phi)|_{\Phi^* g}^2 \, \mathrm{d}v_{\Phi^* g}.$$

□

Proposition 2. *Let*

1. (M, g_1) *be a Riemann surface;*
2. $\Phi : (M, g_1) \to (M, g_2)$ *be a conformal map;*
3. *and* $u \in C^2(M; N)$ *is a harmonic map with respect to* (M, g_2).

Then

$$u \circ \Phi \in C^2(M; N) \text{ is a harmonic map with respect to } (M, g_1).$$

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Proof. By setting $\Phi^* g_2 = e^{2\varphi} g_1$, we have

$$\begin{aligned} E(v\circ\Phi, g_1) &= \frac{1}{2}\int_M tr_{g_1}\left((v\circ\Phi)^* h\right) \mathrm{d}v_{g_1} \\ &= \frac{1}{2}\int_M tr_{e^{-2\varphi}\Phi^* g_2}\left(\Phi^*\left(v^* h\right)\right) e^{-2\varphi} \mathrm{d}v_{\Phi^* g_2} \quad (n = dim\ M = 2) \\ &= \frac{1}{2}\int_M tr_{\Phi^* g_2}\left(\Phi^*\left(v^* h\right)\right) \mathrm{d}v_{\Phi^* g_2} \quad \left((cA)^{-1} = \frac{1}{c}A^{-1}\right) \\ &= \frac{1}{2}\int_M tr_{g_2}\left(v^* h\right) \mathrm{d}v_{g_2} \\ &= E(v, g_2), \end{aligned}$$

for all $v \in C^2(M; N)$. □

Remark 3. 1. *Harmonic maps from S^1 to N correspond to* **closed geodesic** *in N.*

2. *The set of harmonic maps from a Riemannian surface M depends only on the conformal structure of M.*
3. *Let $Id : (M, g) \to (M, g)$ be the identity map. then Id is a harmonic map.*

   Proof. Since $u(x) = Id(x) = x$, we have

   $$\begin{aligned} \tau^k(u) &= g^{\alpha\beta}\left[u^k_{\alpha\beta} - \left(\Gamma^M\right)^{\gamma}_{\alpha\beta} u^k_{\gamma} + \left(\Gamma^N\right)^k_{ij}(u) u^i_{\alpha} u^j_{\beta}\right] \\ &= g^{\alpha\beta}\left[0 - \left(\Gamma^M\right)^{\gamma}_{\alpha\beta}\delta^k_{\gamma} + \left(\Gamma^M\right)^k_{ij}\delta^i_{\alpha}\delta^j_{\beta}\right] \\ &= 0. \end{aligned}$$

   □

4. *For $n = dim\ M = 2$, any conformal map $\Phi : (M, g_1) \to (M, g_2)$ is a harmonic map.*

   Proof.

   $$(M, g_1) \overset{\Phi}{\to} (M, g_2) \overset{Id}{\to} (M, g_2).$$

□

REFERENCES

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