Finite difference calculations of the deformations of multi-diameter workpieces during turning

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Abstract

This paper presents an easily implemented finite difference (FD) analysis method for calculating the deformations of multi-diameter workpieces during turning. Finite difference models have been developed to describe the deformation of a workpiece when the turning conditions such as the cutting force, the workpiece material as well as shape dimensions, and the clamping types are taken into account. The proposed method has been verified by comparing it with the theoretical results of a simple workpiece with uniform diameter. Based on the results of calculation, a modified correction method, which is used to compensate the deflections in order to achieve the required accuracy of workpiece diameter, is also proposed for turning operations. © 2000 Elsevier Science S.A. All rights reserved.

Keywords: Finite difference; Deformation; Accuracy; Turning

1. Introduction

One of the error sources that affects the diameter of a workpiece machined on a lathe is caused by the elastic deformation of the workpiece. It is well known that a slender workpiece, which can be structurally represented as a simple beam, will have a non-uniform final shape from a longitudinal turning process as unequal deflections develop in the radial direction due to cutting force [1–5]. However, when the cross-section of a workpiece changes along its length as shown in Fig. 1, frequently encountered in industry [6,7], the deformations of the workpiece along the longitudinal axis will be more complicated than those of a slender workpiece. Therefore, deformations due to radial cutting force and the final shape of the workpiece are not easily predicted.

In order to achieve a higher accuracy of dimension, prediction of the final shape of workpiece are required to be made in advance. It is obvious that the better the knowledge of the workpiece deformations, the easier and quicker it would be to achieve the required dimension and therefore the less cost. To achieve high machining accuracy, computer controlled error correction systems have been broadly studied [8–11], but the machining accuracy obtained in practice proved not to be entirely satisfactory. The unsatis-

factory results may be explained by lack of sufficient deformation information to cover widely different types of workpieces.

In the present paper, a finite difference method for calculating the workpiece deformations is presented. The FD method for the calculation of workpiece deflections will be particularly useful when the cross-section of the workpiece changes along the longitudinal axis.

2. Mathematical models

A workpiece with a non-uniform diameter structure can be regarded as a non-prismatic beam in mechanics. The workpiece has to be mathematically divided into a series of equally spaced intervals so that within each section a constant moment of inertia is kept. Let the sequence number of the different sections is denoted by i. The joint points between sections can then be described by $\{x[i], \; i=1, 2, \ldots, N\}$. The deflections of workpiece are accordingly denoted by $\{y[i], \; i=1, 2, \ldots, N\}$ (Fig. 2).

If the equation for the deflected curve of the workpiece is taken to be $y=f(x)$, then the first derivation $(\mathrm{d}y/\mathrm{d}x)_{x=x[i]}$ is the slope of the curve at position $x[i]$. Provided the space interval h is small, then the slope can be approximately
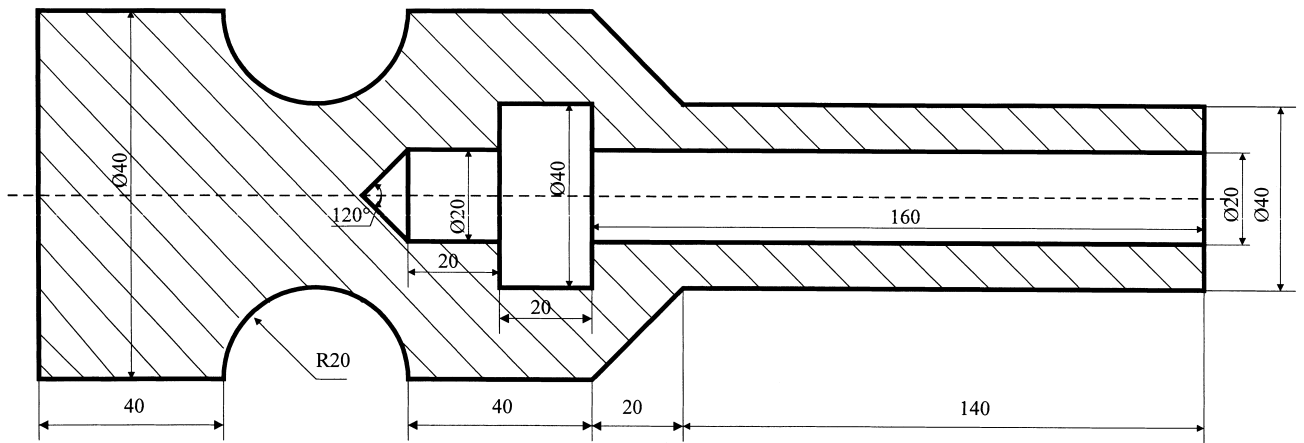
Fig. 1. Diagram for a complicated workpiece (dimensions: mm).

expressed by

$$\left(\frac{dy}{dx}\right)_{x=x[i]} \approx \frac{y[i+1] - y[i-1]}{2h}. \tag{1}$$

The rate of change of the first derivation, i.e. the rate of change of the slope $(d^2y/d^2x)_{x=x[i]}$ is given in the same way approximately as the slope to the right minus the slope to the left at position $x[i]$ divided by the interval between them. Thus

$$\left(\frac{d^2y}{d^2x}\right)_{x=x[i]} = \frac{((y[i+1] - y[i])/h) - ((y[i] - y[i-1])/h)}{h}$$

$$= (1/h^2)(y[i+1] - 2y[i] + y[i-1]). \tag{2}$$

When the second-order derivation for the deflected curve of the workpiece is denoted by $(d^2y/d^2x)_{x=x[i]}$, which is approximately equal to the curvature [12], the curvature

equation for the workpiece can be represented as

$$\frac{M_{[i]}}{E_{[i]}I_{[i]}} = -\left(\frac{d^2y}{d^2x}\right)_{x=x[i]}, \tag{3}$$

where, $M_{[i]}$ is the moment, $E_{[i]}$ the modulus of elasticity of the workpiece material, $I_{[i]}$ is the moment of inertia for the described section depending on the geometry of the cross-section of the workpiece. The workpiece in turning operations is circular in cross-section, and the moment of inertia is given by

$$I_{[i]} = \frac{\pi(D_{[i]}^4 - d_{[i]}^4)}{64}, \tag{4}$$

where, $D_{[i]}$ is the outer diameter and $d_{[i]}$ is the inner diameter at position $x_{[i]}$.

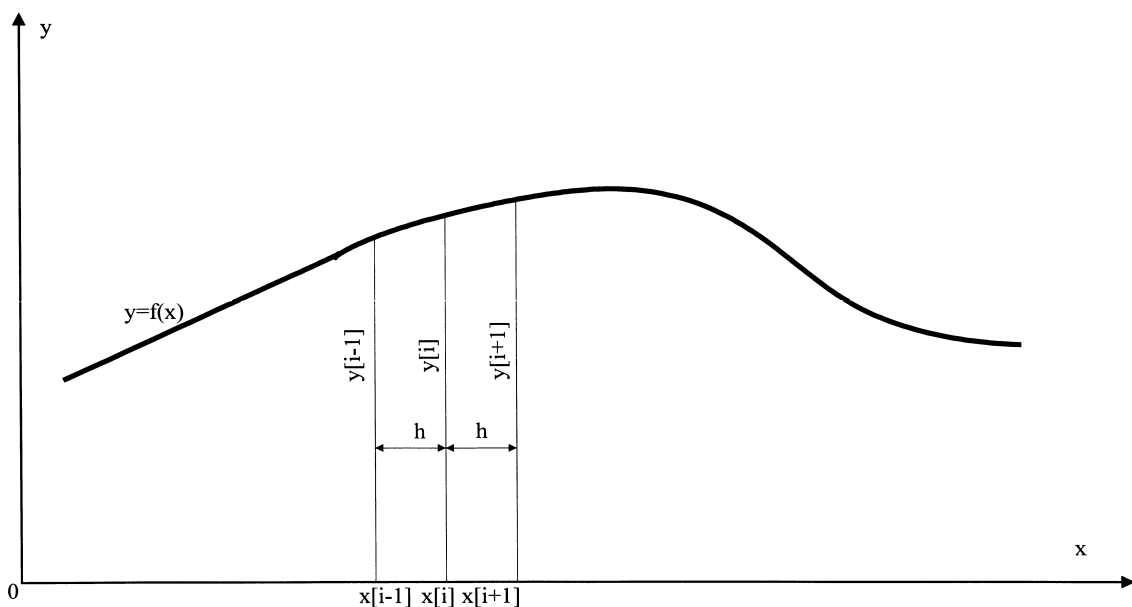Inserting Eq. (2) into Eq. (3) gives the finite difference equation

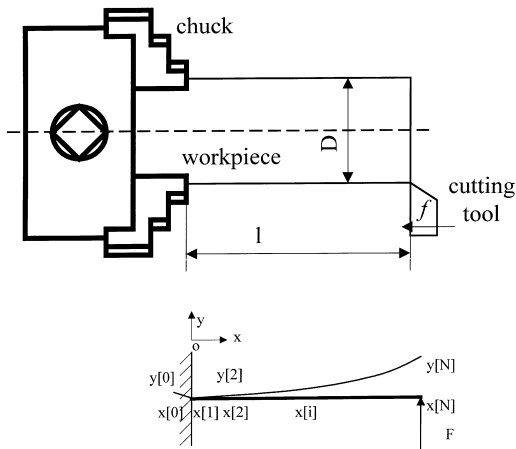Fig. 2. FD analysis for the deflected curve of a workpiece.

Fig. 3. Diagram of a workpiece clamped by a chuck in turning.

$$\left(\frac{M_{[i]}}{E_{[i]}I_{[i]}}\right)_{x=x[i]} = -\frac{1}{h^2}(y[i+1] - 2y[i] + y[i-1]). \tag{5}$$

To use the finite difference mathematical model Eq. (5) to calculate the deformations, it is necessary to know the boundary conditions that are given by clamping types. There are generally three basic types of workpiece clamping:

- workpiece is clamped in a chuck only,
- workpiece is supported between two centers, and
- workpiece is held in the chuck at one end and is simply supported on a center at the tailstock at another end.

3. FD calculations of the deflections of a workpiece clamped in a chuck

Workpiece clamped only in the chuck can be regarded as a cantilever beam (Fig. 3). The maximum deflection occurs at the free end of workpiece where the radial cutting force is acted. The boundary conditions are given that both the deflection and the slope at fixed end point are zero

$$y[1] = 0, \text{ and } \left(\frac{dy}{dx}\right)_{x=x[1]} \approx \frac{y[2] - y[0]}{2h} = 0. \tag{6}$$

When the workpiece is divided into $N-1$ equal intervals, there will be $N-1$ unknown deflections $\{y[i], i=2,3,\ldots,N\}$ to be obtained. The finite difference equation (5) at positions $x[1]$, $x[2]$, ..., and $x[N-1]$ along the workpiece length becomes

$$y[0] - 2y[1] + y[2] = \frac{M_{[1]}h^2}{E_{[1]}I_{[1]}},$$

$$y[1] - 2y[2] + y[3] = \frac{M_{[2]}h^2}{E_{[2]}I_{[2]}}, \tag{7}$$

$$\cdots$$

$$y[N-2] - 2y[N-1] + y[N] = \frac{M_{[N-1]}h^2}{E_{[N-1]}I_{[N-1]}}.$$

When the radial cutting force is acted on the free end of the workpiece, the moments at positions $\{x[i], i=1,2,\ldots,N\}$

are given by

$$M_{[i]} = F(x[i] - l), \tag{8}$$

where, l is the workpiece cutting length.

In the first of Eq. (7), there appears a fictitious deflection $y[0]$ located at an imaginary point to the left of the fixed point (Fig. 3). This fictitious deflection can be expressed in terms of the real deflections by using the second of Eq. (6) for the fixed end point, namely, that the slope is zero. The result is $y[2]=y[0]$. Using the conditions $y[1]=0$ and $y[2]=y[0]$, a solution for any of the deflection values $\{y[i], i=2,3,\ldots,N\}$ can then be obtained by solving the resulting simultaneous equation (7).

To verify the mathematical model, the calculations have been performed for a simple workpiece with length of 200 mm, outer diameter of 40 mm, and inner diameter of zero. The cutting force of 200 N is applied at the free end of the workpiece. The modulus of elasticity is $E=2.0\times10^5$ N/mm^2. The theoretical solution for the workpiece is given by [12]

$$y = \frac{Fx^2}{6EI}(x - 3l). \tag{9}$$

The workpiece is divided into 40 sections with equal spaced interval $h=5$ mm for applying finite difference equation (7). The results from finite difference calculation and the theoretical solution are shown in Table 1. The theoretical solution for the simple workpiece at the free end gives a maximum deflection of 21.2207 μm, and again the FD calculation result is 20.4514 μm, which shows that the results from FD have a very small difference compared with the theoretical results. Obviously, the higher the number of sections divided (the smaller h), the greater the accuracy of solution of FD. Thus, the comparisons give confidence in the FD mathematical model.

The FD method is also applied to calculate the deflections for the workpiece with dimensions shown in Fig. 1. Other conditions are $E=2.0\times10^5$ N/mm^2 and $h=2.5$ mm. The FD calculation results are shown in Fig. 4. It is very difficult to use theoretical method to calculate the deformations for this complicated workpiece. Although it can be solved through finite element method (FEM), FD is easier to implement. FD
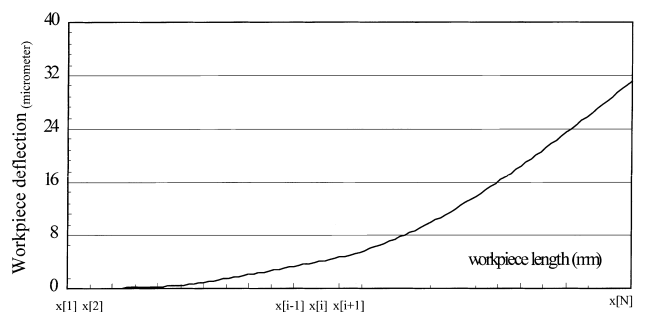

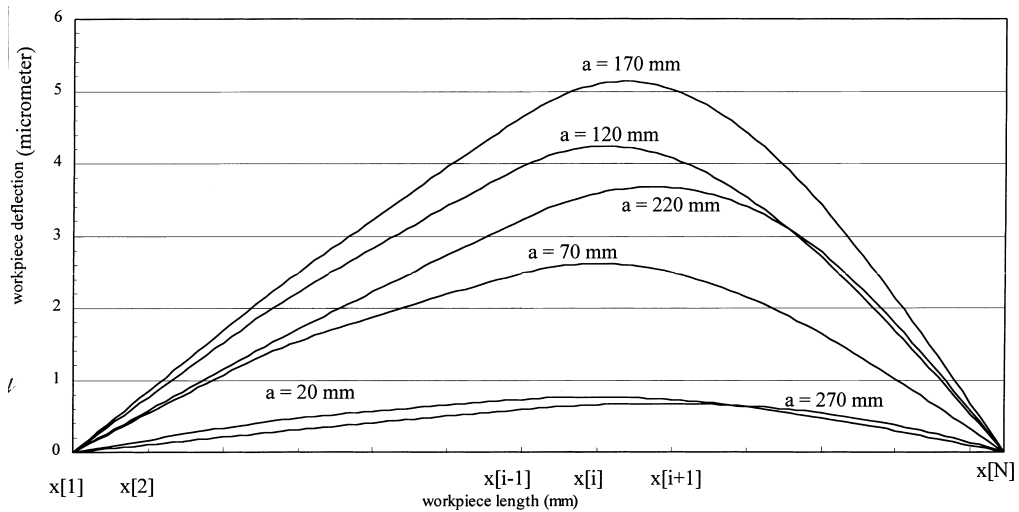Fig. 4. Deflection curve of a workpiece clamped by a chuck in turning.

Table 1
The results from FD compared with the theoretical solution

$X[i]$ (mm)	FD deflection of workpiece (µm)	Theoretical results (µm)	Absolute difference (µm)	Relative difference (%)
0	0	0	0	0
20	0.30239	0.3077	0.005305	1.754396
40	1.16183	1.18836	0.026526	2.283118
60	2.51465	2.57831	0.063662	2.531645
80	4.29719	4.4139	0.116715	2.71608
100	6.44578	6.63146	0.185683	2.880692
120	8.89677	9.16733	0.270567	3.041184
140	11.5865	11.9579	0.37136	3.205112
160	14.4513	14.9394	0.48808	3.377417
180	17.4275	18.0482	0.6207	3.561614
200	20.4514	21.2207	0.76924	3.7613

method is more able to cope with the deflections for wide variety of workpiece shapes.

4. Calculation of the deflections of a workpiece supported between the two centers

A workpiece that is supported between two centers is shown in Fig. 5, and the cutting force is acted on the intermediate of the workpiece $x[i]=a$. The workpiece itself has no deflections at the supporting two centers, so the boundary conditions are given by

$$y[1] = 0, \text{ and } y[N] = 0. \tag{10}$$

The moment for a workpiece that is supported between centers while the cutting force is acted on the intermediate of the workpiece is expressed as

$$M_{[i]} = \begin{cases} \dfrac{F(l-a)x[i]}{l}, & (0 \le x[i] < a), \\ \dfrac{Fa}{l}(l - x[i]), & (a \le x[i] \le l). \end{cases} \tag{11}$$

The finite difference equation (5) can be rearranged as

$$y[i-1] - 2y[i] + y[i+1] = -h^2\left(\frac{M_{[i]}}{E_{[i]}I_{[i]}}\right)_{x=x[i]},$$

$$(i = 2, 3, \ldots, N-1). \tag{12}$$

Fig. 5. Diagram of a workpiece supported between two centers.

Let $-h^2\left(M_{[i]}/E_{[i]}I_{[i]}\right)_{x=x[i]} = C[i], \quad (i = 2, 3, \ldots, N-1)$, the series of simultaneous equation (13) is then expressed as

$$\begin{vmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{vmatrix} \left\{ \begin{array}{c} y[2] \\ y[3] \\ \vdots \\ y[N-1] \end{array} \right\} = \left\{ \begin{array}{c} C[2] \\ C[3] \\ \vdots \\ C[N-1] \end{array} \right\}. \tag{13}$$

Applying Gauss elimination method to the series of simultaneous equation (13)

$$\begin{vmatrix} 1 & -1/2 & & & & \\ & 1 & -2/3 & & 0 & \\ & & \ddots & \ddots & & \\ & & & 1 & -(i-1)/i & \\ 0 & & & & \ddots & \ddots \\ & & & & & 1 \end{vmatrix} \left\{ \begin{array}{c} y[2] \\ y[3] \\ \vdots \\ y[i] \\ \vdots \\ y[N-1] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} D[2] \\ D[3] \\ \vdots \\ D[i] \\ \vdots \\ D[N-1] \end{array} \right\}, \tag{14}$$

where, $D[2], D[3], \ldots, D[i], \ldots, D[N-1]$ are constants after Gauss elimination

$$D[2] = -\frac{1}{2}C[2], \qquad D[i] = -\frac{i-1}{i}(C[i] - D[i-1]),$$

$$(i = 3, 4 \ldots, N-1). \tag{15}$$

Then, a solution for deflection values $\{y[i], i=2,3,\ldots,N\}$ can be obtained by solving the resulting simultaneous equation (14) and using the boundary conditions (10).

Fig. 6. Deflection curves of a workpiece held between two centers.

The results are

$$
\begin{aligned}
& y[N] = 0, \\
& y[N-1] = D[N-1], \\
& y[i] = D[i] + \frac{i-1}{i} y[i+1], \quad (i = N-2, N-3, \ldots, 2), \\
& y[1] = 0.
\end{aligned}
$$

$$(16)$$

However, the deflections of workpiece depend on the position $x[i]=a$, where the cutting force is acted. When cutting tool feeds along the length of the workpiece, i.e. the cutting points $x[i]=a$ that are from l to zero, there will be a variation in deflections over the length. Trying different a, the maximum displacement of the workpiece can be obtained. From Eq. (11) to Eq. (16), the deflection shape curve $y[i]$ can be expressed as

$$
y[i] = f(y[i], a)|\{x = a\}, \quad a \in [0, l]. \tag{17}
$$

When the workpiece in Fig. 1 is supported between two centers, the deflection shape curve for the workpiece is shown in Fig. 6. Other conditions are given by: cutting force applied at $a=20, 70, 120, 170, 220, 270$ mm, respectively, and $h=2.5$ mm, $E=2.0\times10^5$ N/mm^2.

5. Calculation of the deflections of a workpiece held between the chuck and the tailstock

The workpiece that is held in the chuck at one end and another is simply supported on the center at the tailstock can be regarded as a beam with one fixed end and another the simply supported end (Fig. 7). The cutting force is applied to the intermediate of the workpiece at position $x[i]=a$. The boundary conditions for this kind of workpiece clamping method are also given by Eq. (10).

The moment is expressed as

$$
M_{[i]} = \begin{cases}
\dfrac{F(l-a)}{2l^3}\{(l-a)^2 l - l^3 + x[i][(3l^2 - (l-a)^2)]\}, & (0 \le x[i] < a), \\[2mm]
\dfrac{Fa}{2l^3}(3l^2 - 3lx[i] - al + ax[i]), & (a \le x[i] \le l).
\end{cases}
$$

$$(18)$$

The remaining analysis is the same as for the workpiece which is supported between two centers. A solution for deflection values $\{y[i], i=2,3,\ldots,N\}$ can be obtained by solving Eqs. (18), (12), (13) and (14) and using the boundary conditions (10).

The workpiece has the same shape dimensions and material as in Section 4 but it is fixedly held in the chuck at one end, and another is simply supported on the center at the tailstock, its deflection shape curve is shown in Fig. 8.

6. Discussion and recommendations

It can be seen from Fig. 6 to Fig. 8 that the deflection shape curve of the workpiece is sensitive to the clamping
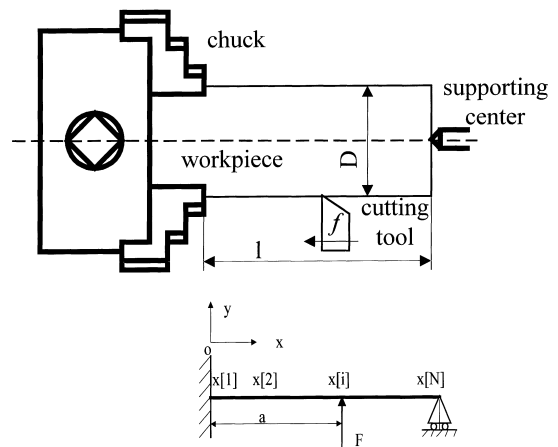

Fig. 7. Diagram of a workpiece held between a chuck and a tailstock.
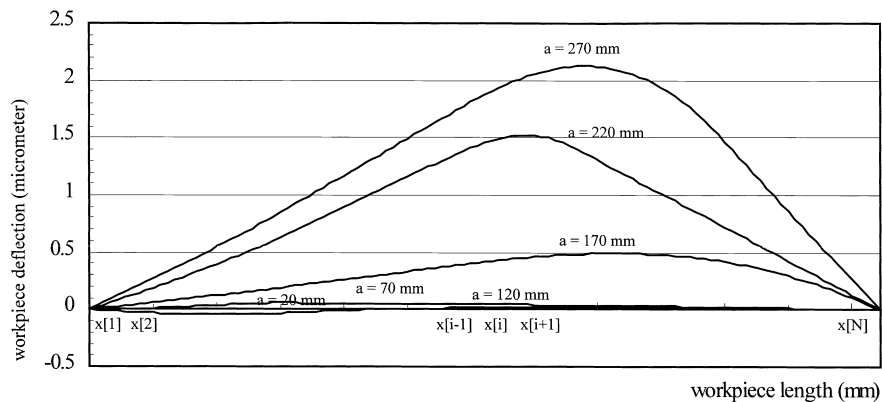
Fig. 8. Deflection curves of a workpiece held between a chuck and a tailstock.

methods. It yields far bigger deflections when a workpiece is held in a chuck only than when it is held between a chuck and a tailstock as well as between two centers. This is more important for workpieces with a greater length-to-diameter ratio. However, such results are obtained on the assumption that the clamping and supporting devices are absolutely rigid. It is virtually impossible to achieve this absolute rigidity. In practice, the deflections for the headstock, tailstock, and supporting centers from the reaction of the cutting force should be considered. The diameter errors due to workpiece deflection under the cutting force in turning depend on the combined rigidity of the machine–tool–workpiece system and the clamping devices [1–3,5,13].

Once the errors are determined, some means for the error corrections must be developed to improve the accuracy of the machined workpiece. To eliminate the workpiece errors, the common and easy way on CNC machining centers is the workpiece program modifications. The nominal depth of cut $a_{p \text{ nominal}}$ is modified in turning operations. The modified depth of cut $a_{p \text{ modified}}$ in turning process is the sum of the nominal depth of cut and the workpiece prediction deflections $y[i]$ at every cutting point [14]. Since in practice, the data for the CNC turning are introduced through the desired final diameters D_{desired}, the corrections can be achieved through programming the lathe to modified final diameters

D_{modified} by subtracting twice the modified depth of cut from the diameter before cutting (Fig. 9):

$$D_{\text{modified}} = D_{\text{desired}} - a_{p \text{ modified}} = D_{\text{desired}} - 2[a_{p \text{ nominal}} + y[i]]. \tag{19}$$

The proposed correction method decreases the diameter errors caused by workpiece deflection. However, deflection of workpiece is not the only source of machining errors. Wears in cutting tools, thermal deformations and geometric errors of machine tools, error transmission effects due to variations in the cutting force along the workpiece caused by variations in dimensions of cut, and vibrations of the machine–tool–workpiece system contribute to diameter errors in turning, too. These causes should be considered for a comprehensive solution to achieve the high precision of workpiece.

7. Conclusions

It is very difficult to use theoretical method to calculate the deformations for the workpieces with non-uniform cross-section areas. This paper presented an easily implemented finite difference (FD) analysis method to predict the deformations of multi-diameter workpieces during turning operations. The results from FD calculations compare well with the theoretical results for a simple workpiece with uniform diameter, which gives confidence of the FD analysis method. Finite difference models have been developed to describe the deformation of a workpiece with complicate geometries that is clamped only in chuck, held between chuck and tailstock, or held between supporting centers. The cutting force, the workpiece material as well as shape parameters are also taken into account. The results from the FD calculation show that the workpiece deflection is very sensitive to the clamping methods when the clamping and supporting device are far more rigid. Based on the results of calculation, a modified correction method to compensate for the deflections is presented for turning operations.
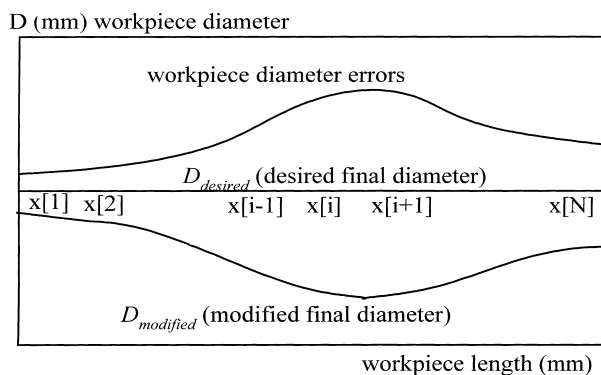

Fig. 9. The modified diameter in the correction program.

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