电动力学中的电荷禁闭

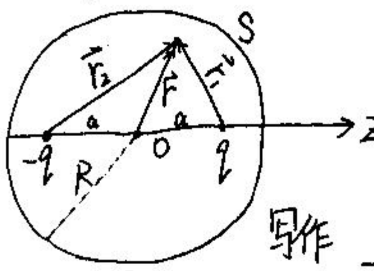
提出背景——夸克禁闭 不可交换规范理论

§1 线性介质中的电荷禁闭

区域（电荷被禁闭，外界对内部电荷无觉察）

最简单的区域—球域。$(a<R)$ 若球域分界面 S 上无电力线穿出，要求

$$E_n|_S=0 \qquad (1.1)$$



作为边界条件。E_n 为场强法向分量。欲求满足该条件的静电势函数，可归结为解球形区域的静电 Poisson 方程，源为两个 δ 函数。Gauss 单位制里写作

$$\nabla^2\phi(\vec{r})=-4\pi q[\delta(\vec{r}-\vec{a})-\delta(\vec{r}+\vec{a})] \qquad (1.2)$$

该方程是线性方程 $\quad \phi=\phi_1+\phi_2+\phi_3$

ϕ_1,ϕ_2 分别为正、负 δ 函数为源的方程特解，ϕ_3 为 Laplace 方程通解

$$\begin{cases}\nabla^2\phi_1=-4\pi q\delta(\vec{r}-\vec{a})\\ \nabla^2\phi_2=4\pi q\delta(\vec{r}+\vec{a})\\ \nabla^2\phi_3=0\end{cases}$$

解得
$$\begin{cases}\phi_1=\dfrac{q}{r_1}\\ \phi_2=-\dfrac{q}{r_2}\\ \phi_3=\sum_{l=0}^{\infty}C_l r^l P_l(\cos\theta)\end{cases}$$
C_l 由边界条件而定。 (1.3)

$$(1.1)\Rightarrow \qquad \left.\frac{\partial\phi}{\partial r}\right|_{r=R}=0 \qquad (1.4)$$

√利用展式（后面还会使用）
$$\frac{1}{|\vec{r}-\vec{a}|}=\begin{cases}\sum_{l=0}^{\infty}\dfrac{a^l}{r^{l+1}}P_l(\cos\theta) & r>a\\ \sum_{l=0}^{\infty}\dfrac{r^l}{a^{l+1}}P_l(\cos\theta) & r<a\end{cases}$$

可在分界面 $r=R$ 附近，把 ϕ_1 和 ϕ_2 展开。（$r>a$，用展式上部分）

$$\begin{cases}\phi_1(\vec{r})=\dfrac{q}{|\vec{r}-\vec{a}|}=q\sum_{l=0}^{\infty}\dfrac{a^l}{r^{l+1}}P_l(\cos\theta)\\ \phi_2(\vec{r})=\dfrac{-q}{|\vec{r}+\vec{a}|}=-q\sum_{l=0}^{\infty}\dfrac{(-a)^l}{r^{l+1}}P_l(\cos\theta)\end{cases} \qquad (1.5)$$

与 (1.3) 式代入边界条件 (1.4)，得

$$\left.\sum_{l=0}^{\infty}\left\{-(l+1)[1-(-1)^l]\frac{qa^l}{r^{l+2}}+lC_l r^{l-1}\right\}P_l(\cos\theta)\right|_{r=R}=0$$

$$C_l=q\frac{l+1}{l}[1-(-1)^l]\frac{a^l}{R^{2l+1}} \qquad l=1,2,\cdots\infty \qquad C_0 \text{ 任意（取 0）}$$

解为 $\displaystyle \phi=\frac{q}{r_1}-\frac{q}{r_2}+q\sum_{l=1}^{\infty}\frac{l+1}{l}[1-(-1)^l]\frac{a^l r^l}{R^{2l+1}}P_l(\cos\theta)$

改写为 $\phi=\phi_+ +\phi_-$

√
$$\begin{cases}\phi_+=\dfrac{q}{r_1}+q\sum_{l=0}^{\infty}\dfrac{l+1}{l}\dfrac{a^l}{R^{2l+1}}r^lP_l(\cos\theta) & \text{（会与后式比较）}\\ \phi_-=\dfrac{-q}{r_2}-q\sum_{l=0}^{\infty}\dfrac{l+1}{l}\dfrac{(-a)^l}{R^{2l+1}}r^lP_l(\cos\theta)\end{cases} \qquad (1.6)$$

△介质观点

一个介电常数为 ε_1，半径为 R 的电介质球置于介电常数为 ε_2 的电介质内。若在离球心距 a 处 $(a<R)$ 置一点电荷 q，则球内电势为

$$\phi=\frac{q}{\varepsilon_1 r_1}+q\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1}\sum_{l=0}^{\infty}\frac{l+1}{l+\varepsilon_2(l+1)}\frac{a^l r^l}{R^{2l+1}}P_l(\cos\theta),\quad r\leq R \qquad (1.7)$$

当 $\varepsilon_1=1$，$\varepsilon_2=0$ 时与 (1.6) 式中 ϕ_+ 完全相同形式。$q\to-q$，$a\to-a$ 可得 ϕ_-。可见介电常数为 0 的电介质有禁止电力线进入自身的性质，该电介质可造成电荷禁闭。称为"抗电介质"。自然界由"抗色介质"构成的，色场被其阻止外露。

△镜象

改写 $\phi_+=\frac{q}{r_1}+\sum_{l=1}^{\infty}\frac{qa^l}{R^{2l+1}}r^lP_l(\cos\theta)+\sum_{l=1}^{\infty}\frac{1}{l}\frac{qa^lr^l}{R^{2l+1}}P_l(\cos\theta)$.

利用参数积分公式 $\int_0^{\infty}e^{-zl}dz=\frac{1}{l}$

$$\phi_+=\frac{q}{r_1}+\sum_{l=1}^{\infty}\left(\frac{R}{a}q\right)\frac{r^l}{\left(\frac{R^2}{a}\right)^{l+1}}P_l(\cos\theta)+\sum_{l=1}^{\infty}\int_0^{\infty}dz\left(\frac{R}{a}e^{z}q\right)\frac{r^l}{\left(\frac{R^2}{a}e^{z}\right)^{l+1}}P_l(\cos\theta)$$

令 $\vec{a}'=\vec{e}_z\frac{R^2}{a}$，$q'=\frac{R}{a}q$ 则后两项是 $\frac{q'}{|\vec{r}-\vec{a}'|}$ 和 $\frac{e^zq'}{|\vec{r}-e^z\vec{a}'|}$ 分别在 $r>a'$ 和 $r>a'e^z$ 展开

$$\phi_+=\frac{q}{r_1}+\frac{q'}{|\vec{r}-\vec{a}'|}+\int_0^{\infty}dz\frac{e^zq'}{|\vec{r}-e^z\vec{a}'|} \qquad (1.8)$$

可见 ϕ_+ 是由置于 $\vec{a}$ 处的 q 及由 $\vec{a}$ 的共轭点 $\vec{a}'$ 开始向 Z 方向规则排布着的镜象电荷线所形成的静电势。ϕ_- 同理。

※在一条直线上，按指数规则排布着向正、负方向无穷延伸的正负电荷镜象线也可以产生自由 $\pm q$ 电荷对的禁闭效果。即规则排布的电荷线系可形成球状的电荷禁闭区域。

§2 轴对称非线性介质静电禁闭

当 ε 不是常数，是 E 的标量函数时，方程复杂（非线性的）。

若电场的源是孤立的点电荷系，在无源区域，静电方程

$$\begin{cases}\nabla\times\vec{E}=0 & (1.9)\\ \nabla\cdot\vec{D}=0 & (1.10)\end{cases}$$

各向同性介质中 $\vec{D}=\varepsilon(E)\vec{E}$，其中 $E=|\vec{E}|$ (1.11)

H. Lehmann 和 T.T. Wu 讨论电力线禁闭提出系统处于存在轴对称性时的求解改变边界条件（适当地），具有普遍意义。

选用柱坐标系

$$\frac{\partial E_\rho}{\partial z}-\frac{\partial E_z}{\partial\rho}=0 \qquad (1.12)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho)+\frac{\partial D_z}{\partial z}=0 \qquad (1.13)$$

引入标量变量 ψ，与 $\vec{D}$ 存在

留着!! $\Rightarrow$ $2\pi\rho D=\left[(\partial_\rho\psi)^2+(\partial_z\psi)^2\right]^{\frac{1}{2}}$ (1.14※)

$$\rho D_z=\frac{1}{2\pi}\partial_\rho\psi,\quad \rho D_\rho=-\frac{1}{2\pi}\partial_z\psi \qquad (1.14)$$

假定 $E=f(D)$ $\quad \varepsilon=\frac{D}{f(D)}$ $\quad D=(D_\rho^2+D_z^2)^{1/2}$

$$\therefore E_\rho=\frac{f(D)}{D}D_\rho \qquad E_z=\frac{f(D)}{D}D_z \qquad (1.15)$$

将(1.14) (1.15) 代入(1.12)

$$\partial_z \left\{ \frac{f(D)\,\partial_z\psi}{2\pi\rho D} \right\} + \partial_\rho \left\{ \frac{f(D)\,\partial_\rho\psi}{2\pi\rho D} \right\} = 0 \quad (1.16)$$

由(1.14*) 代换 $2\pi\rho D$

$$2\pi\rho D = [(\partial_\rho\psi)^2+(\partial_z\psi)^2]^{\frac{1}{2}}$$

(1.17)

或计算微商 并 $f' = \frac{df(D)}{dD}$ $\quad g = \frac{f(D)}{f'(D)} - D$

得 $$\frac{g}{D}(\partial_z\psi\,\partial_z D + \partial_\rho\psi\,\partial_\rho D) - (g+D)\left[\partial_{zz}\psi + \partial_{\rho\rho}\psi - \frac{\partial_\rho\psi}{\rho}\right] = 0$$

由 $2\pi\rho D = [(\ )^2+(\ )^2]^{1/2}$ $\quad$ (1.14*)

$$\begin{cases} \partial_z D = \frac{1}{4\pi^2\rho^2 D}[\partial_z\psi\cdot\partial_{zz}\psi + \partial_{z\rho}\psi\cdot\partial_\rho\psi] \\ \partial_\rho D = \frac{1}{4\pi^2\rho^2 D}[\partial_\rho\psi\cdot\partial_{\rho\rho}\psi + \partial_z\psi\,\partial_{\rho z}\psi] - \frac{D}{\rho} \end{cases}$$

代入 再两端同乘 $4\pi^2\rho^2 D$

$$*\left[(\partial_z\psi)^2 + \left(1+\frac{g}{D}\right)(\partial_\rho\psi)^2\right]\partial_{zz}\psi + \left[\left(1+\frac{g}{D}\right)(\partial_z\psi)^2 + (\partial_\rho\psi)^2\right]\partial_{\rho\rho}\psi$$
$$-2\frac{g}{D}\partial_z\psi\cdot\partial_\rho\psi\,\partial_{\rho z}\psi - \frac{1}{\rho}\partial_\rho\psi\cdot[(\partial_z\psi)^2+(\partial_\rho\psi)^2] = 0 \quad (1.18)$$

从另一个角度 用势(或流量函数 ψ)表达. 目的 更易推广到轴对称稳态情况的真空爱因斯坦一恩斯特方程.

令 $\vec{E} = -\nabla w$ $\qquad \nabla\cdot\vec{D} = \nabla\cdot[\varepsilon(E)\vec{E}] = 0$ $\qquad$ (1.19)

只考虑轴对称情况. $\vec{E} = \vec{E}(\rho, z)$ $\quad \vec{D} = \vec{D}(\rho, z)$. —— 纬向单位矢量.

由于可验证 对任意函数 $\Omega = \Omega(\rho, z)$ 成立: $\nabla\cdot\left(\frac{\hat{\varphi}\times\nabla\Omega}{\rho}\right) = 0.$ $\quad$ (1.20)

对比 $\vec{D}$ 与 $\rho^{-1}\hat{\varphi}\times\Delta\Omega$ 只差一个无散矢量或常数 且 Ω 任意.

可选择 $\qquad \vec{D} = -\varepsilon(E)\nabla w = \rho^{-1}\hat{\varphi}\times\nabla\Omega.$ $\qquad$ (1.21)

*引入 $\qquad \varepsilon(E) = \rho^{-2} f^2(\vec{E}(\rho, z), \rho, z)$ $\qquad$ (1.22)

得 $\qquad \rho^{-1}\nabla w = -\hat{\varphi}\times\nabla\Omega\cdot f^{-2}$

两边作 $\hat{\varphi}$ 的叉积 $\qquad \rho^{-1}\hat{\varphi}\times\nabla w = -f^{-2}\hat{\varphi}\times(\hat{\varphi}\times\nabla\Omega) = f^{-2}\nabla\Omega$

再两边取散度 $w \Rightarrow \Omega$. $\qquad \nabla\cdot(f^{-2}\nabla\Omega) = 0$ $\qquad$ (1.23)

即 $\qquad f\nabla^2\Omega = 2\nabla f\cdot\nabla\Omega.$ $\qquad$ —— Ernst 方程组的一个. $\quad$ (1.24)

在静电学中, $f \sim \varepsilon$, $\Omega \sim \vec{E}$ 但不能从方程决定 f。广义相对论中存在决定 f 的另一方程 (Ernst)。广义相对论中的 f ~ 度规 g_{00} 有关, 从某种意义上说 g_{00} ~ 等效介电常数.

轴对称 非线性静电学问题

类似

关于类似 给一个演示解性例子(体现静电问题与Ernst方程最简单解的相似性) 4

将 $f\nabla^2\Omega=2\nabla f\cdot\nabla\Omega$ 用柱坐标表示，用 ε 代替 f 可得

$$* \quad \frac{\partial^2\Omega}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\Omega}{\partial\rho}+\frac{\partial^2\Omega}{\partial z^2}-\left(\frac{\partial\ln\varepsilon}{\partial\rho}\frac{\partial\Omega}{\partial\rho}+\frac{\partial\ln\varepsilon}{\partial z}\frac{\partial\Omega}{\partial z}\right)=0. \qquad (1.25)$$

若 ε 直接依赖于 ρ, z 例如 ε 等于常数 则上式变为

$$\frac{\partial^2\Omega}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\Omega}{\partial\rho}+\frac{\partial^2\Omega}{\partial z^2}=0.$$ 可用Fourier变换直接求出满足一定边条件解

若令 $\Omega=f^2$，则 $\nabla(f^{-2}\nabla\Omega)=0 \Rightarrow \nabla\left(\frac{\nabla\Omega}{\Omega}\right)=0$. 即 $\nabla^2\ln\Omega=0$.

显然 $\ln\Omega$ 是调和函数（满足Laplace方程）. 故有

$$\Omega=Ae^{\sigma(\rho,z)}, \quad \nabla^2\sigma=0. \qquad \cdots \text{式} \qquad (1.26)$$

代入到 $\varepsilon=\rho^{-2}\Omega(\rho,z)$.

$$\varepsilon=A\cdot\rho^{-2}e^{\sigma(\rho,z)}, \quad \nabla^2\sigma=0. \qquad (1.27)$$

(1.26)，(1.27) 给出 (1.25) 自洽解

也可给出 $\vec{E}$ 的自洽解

$$\vec{E}A\rho^{-2}e^{\sigma(\rho,z)}=\rho^{-1}\hat{\varphi}\times[\nabla\sigma(\rho,z)]e^{\sigma(\rho,z)}.$$

$$\therefore \vec{E}=C\cdot\rho\cdot\underbrace{\hat{\varphi}\times[\nabla\sigma(\rho,z)]}_{=\ \hat{\rho}\frac{\partial\sigma}{\partial z}-\hat{z}\frac{\partial\sigma}{\partial\rho}} \qquad \begin{cases} E_\rho=C_\rho\cdot\rho\frac{\partial\sigma}{\partial z} \\ E_z=C_z\cdot\rho\frac{\partial\sigma}{\partial\rho}\end{cases} \quad \nabla^2\sigma(\rho,z)=0$$

$(1.18) \xLeftrightarrow{\text{类似}} (1.25)$.

讨论 $\varepsilon=E^{\sigma}$ $(E=|\vec{E}|)$ 的情况. σ 为任意实数.

与(1.22) $\varepsilon=\rho^{-2}f^2(\vec{E}(\rho,z),\rho,z)$ 比较. 且 $\vec{D}=-\varepsilon(E)\nabla\omega=\rho^{-1}\hat{\varphi}\times\nabla\Omega$

得 $E=\rho^{-1}\varepsilon^{-1}|\nabla\Omega|$ $\quad D=E^{\sigma+1}=\rho^{-1}|\nabla\Omega|$

对 $E^{\sigma+1}=\rho^{-1}|\nabla\Omega|$ 两端求微商

$$\begin{cases}\frac{\partial\ln\varepsilon}{\partial\rho}=\frac{\sigma}{1+\sigma}\left[\partial_\rho\ln|\nabla\Omega|-\frac{1}{\rho}\right] \\ \frac{\partial\ln\varepsilon}{\partial z}=\frac{\sigma}{1+\sigma}\left[\partial_z\ln|\nabla\Omega|\right]\end{cases}$$

$$(1.25)\Rightarrow\frac{\partial^2\Omega}{\partial\rho^2}-\frac{1}{\rho}\left(\frac{1}{1+\sigma}\right)\frac{\partial\Omega}{\partial\rho}+\frac{\partial^2\Omega}{\partial z^2}-\left(\frac{\sigma}{1+\sigma}\right)\cdot\left\{\frac{\partial\Omega}{\partial\rho}\frac{\partial\ln|\nabla\Omega|}{\partial\rho}+\frac{\partial\Omega}{\partial z}\frac{\partial\ln|\nabla\Omega|}{\partial z}\right\}=0$$

$|\nabla\Omega|=[(\partial_\rho\Omega)^2+(\partial_z\Omega)^2]^{1/2}$，计算 $\partial_\rho\ln|\nabla\Omega|$ $\partial_z\ln|\nabla\Omega|$. (1.28)

$$[(\partial_\rho\Omega)^2+(\partial_z\Omega)^2]\left\{\partial_{\rho\rho}\Omega-\frac{1}{\rho}\partial_\rho\Omega+\partial_{zz}\Omega\right\}+\sigma\left\{(\partial_\rho\Omega)^2\partial_{zz}\Omega+(\partial_z\Omega)^2\partial_{\rho\rho}\Omega-2\partial_z\Omega\cdot\partial_\rho\Omega\cdot\partial_{\rho z}\Omega\right\}=0$$

令 $\sigma=\frac{g}{D}$ ~~$\psi=\Omega$~~. 上式 $\Rightarrow$ (1.18). 精确至任意常数 $\psi=\Omega$.

推导(1.18) 简单形式. 回忆 $\varepsilon=\frac{D}{f(D)}$ $\quad \varepsilon=\rho^{-2}\Omega(\rho,z)$ 一致

$\Omega=\rho^2\frac{D}{f(D)}$. 代入到 $\nabla\left(\frac{\nabla\Omega}{\Omega}\right)=0$. 令 $\Omega=\frac{\psi}{2\pi}$.

得关于 ψ 的方程. $\nabla\cdot\left(\frac{\nabla\psi}{\rho^2 D}\cdot f(D)\right)=0$. (1.29) 更简洁. 可能更有用

§3 吴大峻的静电禁闭理论

讨论电荷禁闭，设电荷被禁闭在区域 B 内，电力线不能从 B 的界面穿出。

边界条件 $\quad \psi|_B = 0 \quad \nabla\psi|_B = 0 \quad$ (过定的边界条件) $\quad (30)$

在几种简化条件下 明显解出 ψ 及边界形状是可行的。

1. 当 $D \to \infty$，$\frac{g}{D} \to 0$ 时的渐近形行为

代入到 (1.18) 式 得 $\quad \partial_{\rho\rho}\psi + \partial_{zz}\psi - \frac{1}{\rho}\partial_\rho\psi = 0$

记作 $\quad \nabla^2\psi - \frac{2}{\rho}\partial_\rho\psi = 0 \quad (1.31)$

边界条件 设 R 为区域 B 线度

$$\begin{cases} \psi = 0 \quad \nabla\psi = 0 & B\text{边界上} \\ \psi(0,z) = \begin{cases} Q\ ? & |z|<R \\ 0 & |z|>R \end{cases} \\ \psi \to 0 & z^2+\rho^2 \to \infty \end{cases}$$

解 $\psi = -\frac{1}{2}Q\left\{\frac{z-R}{[(z-R)^2+\rho^2]^{1/2}} - \frac{z+R}{[(z+R)^2+\rho^2]^{1/2}}\right\}$

对于小 R ($z, \rho \gg R$) 引入 $\quad r = \sqrt{z^2+\rho^2} \quad r \gg R$

级数展开 精确至 $\frac{R}{r}$ 的一次项。

$$\frac{z \pm R}{[(z\pm R)^2+\rho^2]^{1/2}} \approx \frac{z}{r} \pm \frac{R}{r} \mp \frac{z^2}{r^3}R$$

$$\frac{z+R}{[(z+R)^2+\rho^2]^{1/2}} = \frac{z+R}{[R^2+z^2-2Rz+\rho^2]^{1/2}} = \frac{z+R}{[r^2-2Rz]^{1/2}}$$

$R=0$ 处展开

$$\frac{z}{r} + \frac{(r^2-2Rz)^{1/2} + (z+R)z\frac{1}{\sqrt{r^2-2Rz}}}{(r^2-2Rz)} \qquad \frac{r^2-2Rz+z^2+Rz}{(r^2-2Rz)^{3/2}}$$

$$\frac{r^2-z^2}{r^3}R \qquad \frac{R}{r} - \frac{z^2}{r^3}R$$

$$\psi \approx \frac{QR\rho^2}{(\rho^2+z^2)^{3/2}}$$

$$D^2 = D_\rho^2 + D_z^2 = \frac{Q^2R^2}{4\pi^2}\frac{4z^2+\rho^2}{(z^2+\rho^2)^4}$$

2. 当禁闭线度 R 很大，正负电荷甚远时，与解线性问题相类似。认为 D 与 E 相差不大

把 $E(D) = f(D)$ 在 $E=D$ 附近展开 $\quad f(D) = E_0 + b_1 D \quad$ 取单位 $E_0 = b_1 = 1$.

$f(D) = 1 + D \quad g = 1$. 可用叠代法解方程，只讨论最简单的零级解 $\psi^{(0)}$.

$$\partial_\rho\left[\frac{\partial_\rho\psi}{\rho}\left(1+\frac{1}{D}\right)\right] + \partial_z\left[\frac{\partial_z\psi}{\rho}\left(1+\frac{1}{D}\right)\right] = 0$$

变量代换 $\quad \rho = \rho'\sqrt{R} \qquad z = z'R$

$$D = \frac{1}{2\pi\rho}\sqrt{(\partial_\rho\psi)^2+(\partial_z\psi)^2}$$

$$= \frac{1}{2\pi\rho'\sqrt{R}}\sqrt{\frac{1}{R}(\partial_{\rho'}\psi)^2+\frac{1}{R^2}(\partial_{z'}\psi)^2} = \frac{1}{2\pi\rho' R}\sqrt{(\partial_{\rho'}\psi)^2+\frac{1}{R}(\partial_{z'}\psi)^2}$$

R 很大时 取 $\frac{1}{R}$ 级数展开

$$\frac{1}{D} \approx -\frac{2\pi\rho' R}{\partial_{\rho'}\psi}\left[1-\frac{1}{2R}\left(\frac{\partial_{z'}\psi}{\partial_{\rho'}\psi}\right)^2\right] \qquad \text{其中取 } \sqrt{(\partial_{\rho'}\psi)^2} = -\partial_{\rho'}\psi$$

对方程也做代换 $\quad z \to z' \quad \rho \to \rho' \quad$ 代入 $\frac{1}{D}$ 得

$$\partial_{\rho'}\left[\frac{\partial_{\rho'}\psi}{\rho'} - 2\pi R + \pi\left(\frac{\partial_{z'}\psi}{\partial_{\rho'}\psi}\right)^2\right] + \frac{\partial_{z'}}{R}\left[\frac{\partial_{z'}\psi}{\rho'} - 2\pi R\frac{\partial_{z'}\psi}{\partial_{\rho'}\psi} + \frac{\pi\partial_{z'}\psi}{\partial_{\rho'}\psi}\left(\frac{\partial_{z'}\psi}{\partial_{\rho'}\psi}\right)^2\right] = 0$$

$$\partial_{\rho'}\left[\frac{\partial_{\rho'}\psi^{(0)}}{\rho'}+\pi\left(\frac{\partial_{z'}\psi^{(0)}}{\partial_{\rho'}\psi^{(0)}}\right)^2\right]-2\pi\,\partial_{z'}\left(\frac{\partial_{z'}\psi^{(0)}}{\partial_{\rho'}\psi^{(0)}}\right)=0.$$

即 $$\partial_{z'}\left(\frac{\partial_{z'}\psi^{(0)}}{\partial_{\rho'}\psi^{(0)}}\right)-\frac{1}{2}\partial_{\rho'}\left[\frac{\partial_{\rho'}\psi^{(0)}}{\pi\rho'}+\left(\frac{\partial_{z'}\psi^{(0)}}{\partial_{\rho'}\psi^{(0)}}\right)^2\right]=0 \qquad (1.32)$$

若 $\rho'=\rho'[z',\psi^{(0)}]$

$$\frac{\partial\rho'}{\partial\psi^{(0)}}=\frac{1}{\partial_{\rho'}\psi^{(0)}} \qquad \frac{\partial\rho'}{\partial z'}=-\frac{\partial_{z'}\psi^{(0)}}{\partial_{\rho'}\psi^{(0)}}$$

对 ρ', z' 的任意函数 $f(\rho',z')$

$$\partial_{\rho'}f=\frac{\partial f}{\partial\psi^{(0)}}\Big/\frac{\partial\rho'}{\partial\psi^{(0)}}$$

$$\partial_{z'}f=\frac{\partial f}{\partial z'}-\frac{\partial f}{\partial\psi^{(0)}}\frac{\partial\rho'}{\partial z'}\Big/\frac{\partial\rho'}{\partial\psi^{(0)}}$$

化为关于 $\rho'(z',\psi^{(0)})$ 的微分方程

$$\frac{\partial\rho'}{\partial\psi^{(0)}}\frac{\partial^2\rho'}{\partial z'^2}+\frac{1}{2\pi}\frac{\partial}{\partial\psi^{(0)}}\left(\rho'\frac{\partial\rho'}{\partial\psi^{(0)}}\right)^{-1}=0$$ 分离变量法求解.

令 $\rho'=X(z')Y(\psi^{(0)})$ 方程化为

$$-X^3X''=\frac{1}{2\pi YY'}\frac{\partial}{\partial\psi^{(0)}}\left(\frac{1}{YY'}\right)$$

解方程并利用边界条件 $$\begin{cases} Y(\psi^{(0)})=\sqrt{\frac{2}{\sqrt{\pi}}}\sqrt{1-\sqrt{\psi^{(0)}}} \\ X(z')=\sqrt{1-z'^2}. \end{cases}$$

于是 $\rho'=\sqrt{1-z'^2}\sqrt{\frac{2}{\sqrt{\pi}}}\sqrt{1-\sqrt{\psi^{(0)}}}$

$$\psi^{(0)}=\left(1-\frac{\sqrt{\pi}}{2}\frac{\rho'^2}{1-z'^2}\right)^2.$$ 将 $\rho'=\rho/\sqrt{R}$ $z'=z/R$ 代入, 得.

$$\psi^{(0)}=\left(1-\frac{\sqrt{\pi}}{2}\frac{\rho^2R}{R^2-z^2}\right)^2 \qquad (1.33)$$

区域边界~~条件~~形状可由 $\psi^{(0)}=0$ 式定出, 是一个椭圆方程.

$$\frac{z^2}{R^2}+\frac{\sqrt{\pi}}{2}\frac{\rho^2}{R}=1$$

零级近似场能量由下式决定

$$V=\int d^3x\int_0^D dD'E(D')=\int d^3x\left(D+\frac{1}{2}D^2\right)$$

柱坐标下 $\int d^3x=\int\rho\,d\rho\,d\varphi\,dz\Rightarrow 2\pi\int_0^{\rho_b(z)}\rho\,d\rho\int_0^R dz$ $\qquad \frac{D^2}{2}$

$$V(R)\sim 4\pi\int_0^{R-\varepsilon}dz\int_0^{\rho_b(z)}\rho\,d\rho\left[-\frac{1}{2\pi\rho}\partial_\rho\psi^{(0)}-\frac{(\partial_z\psi^{(0)})^2}{4\pi\rho\cdot\partial_\rho\psi^{(0)}}+\frac{1}{8\pi^2\rho^2}(\partial_\rho\psi^{(0)})^2\right]$$

以下计算~~积分~~ $V(R)$ $\qquad -\frac{\partial_\rho\psi^{(0)}}{2\pi\rho}\left[1+\frac{1}{2}\left(\frac{\partial_z\psi}{\partial_\rho\psi}\right)^2\right]$

第一项 $$-2\int_0^{R-\varepsilon}dz\int_0^{\rho_b(z)}d\rho\,\partial_\rho\psi^{(0)}$$

$$=-2\int_0^{R-\varepsilon}dz\left[\psi^{(0)}(\rho_b(z),z)-\psi^{(0)}(0,z)\right]$$

边界条件 $\psi(\rho_b(z),z)=\psi^{(0)}(\rho_b(z),z)=0$ $\qquad \psi(0,z)=\psi^{(0)}(0,z)=1$?

考虑两点电荷距离与禁闭区域在 Z 方向上线度同量级，把 Z 轴移向 $z-R=z_1$ 处.
方程不变. $R\to\infty$ 时边界条件变为

$$\psi(0,z)=\begin{cases}0 & z_1>0\\ 1 & z_1<0\end{cases}$$

R 不会出现在边界条件之中. $\psi(\rho,z)$ 存在一个极限. 在特殊情况下, 对于有限 z_1, ρ_b 有限.
对 $\psi^{(0)}=\cdots$ 分别对 ρ, z 取偏商.

$$\partial_\rho\psi^{(0)}=2\left(1-\frac{\sqrt{\pi}}{2}\frac{R\rho^2}{R^2-z^2}\right)\frac{-\sqrt{\pi}\rho R}{R^2-z^2}$$

$$\partial_z\psi^{(0)}=2\left(1-\frac{\sqrt{\pi}}{2}\frac{R\rho^2}{R^2-z^2}\right)\frac{-\sqrt{\pi}Rz\rho^2}{(R^2-z^2)^2}$$

第二项

$$\frac{1}{4\pi}\int_0^{\rho_b(z)}\frac{\left(\partial_z\psi^{(0)}\right)^2}{\partial_\rho\psi^{(0)}}d\rho=\frac{1}{4\pi}\int_0^{\rho_b(z)}2\left(1-\frac{\sqrt{\pi}}{2}\frac{R\rho^2}{R^2-z^2}\right)\frac{-\sqrt{\pi}Rz^2\rho^3}{(R^2-z^2)^3}$$

$$=\frac{-\sqrt{\pi}Rz^2}{2\pi(R^2-z^2)^3}\left[\frac{1}{4}-\frac{\sqrt{\pi}}{12}\frac{R\rho^2}{R^2-z^2}\right]\rho^4\Big|_0^{\rho_b(z)}$$

$$=\frac{-\sqrt{\pi}Rz^2}{2\pi(R^2-z^2)^3}\left[\frac{1}{4}-\frac{\sqrt{\pi}}{6}\frac{R}{R^2-z^2}\frac{2R}{\sqrt{\pi}}\frac{R^2-z^2}{R^2}\right]\frac{4R^2}{\pi}\frac{(R^2-z^2)^2}{R^4}$$

$$=-\frac{z^2}{6\pi\sqrt{\pi}R(R^2-z^2)}$$

第三项

$$\frac{\sqrt{\pi}}{2}\frac{R}{R^2-z^2}\frac{2R}{\sqrt{\pi}}\frac{R^2-z^2}{R^2}$$

$$\frac{1}{8\pi^2}\int_0^{\rho_b(z)}\frac{d\rho}{\rho}\left(\partial_\rho\psi^{(0)}\right)^2$$

$$=\frac{R^2}{4\pi(R^2-z^2)^2}\int_0^{\rho_b(z)}\left(1-\frac{\sqrt{\pi}}{2}\frac{R\rho^2}{R^2-z^2}\right)^2d\rho^2$$

$$=\frac{R^2}{2\pi(R^2-z^2)^2}\frac{-2(R^2-z^2)}{3\sqrt{\pi}R}\left(1-\frac{\sqrt{\pi}}{2}\frac{R\rho^2}{R^2-z^2}\right)^3\Big|_0^{\rho_b(z)}$$

$$=-\frac{R}{6\pi\sqrt{\pi}(R^2-z^2)}\left[0-1\right]=\frac{R}{6\pi\sqrt{\pi}(R^2-z^2)}$$

二.三项合并

$$4\pi\int_0^{R-\varepsilon}\frac{z^2}{6\pi\sqrt{\pi}(R^2-z^2)R}+\frac{R}{6\pi\sqrt{\pi}(R^2-z^2)}\,dz$$

$$\frac{2}{3\sqrt{\pi}}\int_0^{R-\varepsilon}\frac{z^2+R^2}{R(R^2-z^2)}dz=\sim\int_0^{R-\varepsilon}\left(\frac{2R}{R^2-z^2}-\frac{1}{R}\right)=\sim\left[\ln\frac{R+z}{R-z}\Big|_0^{R-\varepsilon}-1\right]$$

$$=\frac{2}{3\sqrt{\pi}}\left[\ln\frac{2R+\varepsilon}{\varepsilon}-1\right]=\frac{2}{3\sqrt{\pi}}\left[\ln R+\ln 2-\ln\varepsilon-1\right]$$

$$V(R)=2(\underset{0}{R-\varepsilon})+\frac{2}{3\sqrt{\pi}}(\ln R-\ln\varepsilon)+O(1)$$

$$V(R)=2R+\frac{2}{3\sqrt{\pi}}\ln R+O(1)$$

3 轴对称方场方程的∂∂变量

$$\rho = \rho(z, \psi)$$

$$\partial_\rho \psi = \frac{1}{\partial_\psi \rho} \qquad \partial_z \psi = -\frac{\partial_z \rho}{\partial_\psi \rho}$$

对 $f = f(\rho, z)$

$$\begin{cases} \partial_\rho f = \frac{\partial f}{\partial \psi} \frac{1}{\partial_\psi \rho} \\ \partial_z f = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \psi} \frac{\partial_z \rho}{\partial_\psi \rho} \end{cases}$$

$$\partial_\rho \left[\frac{f(D)}{D} \frac{\partial_\rho \psi}{\rho} \right] + \partial_z \left[\frac{f(D)}{D} \frac{\partial_z \psi}{\rho} \right] = 0$$

$$\partial_\rho \left[\frac{f(D)}{D} \frac{1}{\rho \partial_\psi \rho} \right] - \partial_z \left[\frac{f(D)}{D} \frac{\partial_z \rho}{\rho \partial_\psi \rho} \right] = 0$$

$$\frac{1}{\partial_\psi \rho} \left[\frac{1}{\rho \partial_\psi \rho} \partial_\psi \left[\frac{f(D)}{D} \right] + \frac{f(D)}{D} \partial_\psi \left(\frac{1}{\rho \partial_\psi \rho} \right) \right] - \frac{\partial_z \rho}{\rho \partial_\psi \rho} \partial_z \left[\frac{f(D)}{D} \right] - \frac{f(D)}{D} \partial_z \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right)$$

$$+ \frac{\partial_z \rho}{\partial_\psi \rho} \left[\frac{\partial_z \rho}{\rho \partial_\psi \rho} \partial_\psi \left[\frac{f(D)}{D} \right] + \frac{f(D)}{D} \partial_\psi \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) \right] = 0$$

$$\frac{f(D)}{D} \left[\frac{1}{\partial_\psi \rho} \partial_\psi \left(\frac{1}{\rho \partial_\psi \rho} \right) + \frac{\partial_z \rho}{\partial_\psi \rho} \partial_\psi \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) - \partial_z \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) \right]$$

$$+ \left[\frac{(\partial_z \rho)^2}{\rho (\partial_\psi \rho)^2} + \frac{1}{\rho (\partial_\psi \rho)^2} \right] \partial_\psi \left(\frac{f(D)}{D} \right) - \frac{\partial_z \rho}{\rho \partial_\psi \rho} \partial_z \left(\frac{f(D)}{D} \right) = 0 \quad *$$

$$D^2 = D_\rho^2 + D_z^2 \qquad D \partial_\psi D = (D_\rho \partial_\rho D_\rho + D_z \partial_\rho D_z) \partial_\psi \rho$$

$$\partial_\psi \left(\frac{f(D)}{D} \right) = \frac{1}{D^2} [D f'(D) - f(D)] \partial_\psi D$$

$$\partial_z \left(\frac{f(D)}{D} \right) = \frac{1}{D^2} [D f'(D) - f(D)] \partial_z D$$

$$\begin{cases} D_\rho = -\frac{1}{2\pi\rho} \partial_z \psi = +\frac{1}{2\pi\rho} \frac{\partial_z \rho}{\partial_\psi \rho} \\ D_z = \frac{1}{2\pi\rho} \partial_\rho \psi = \frac{1}{2\pi\rho} \frac{1}{\partial_\psi \rho} \end{cases}$$

$$D_\rho \partial_\rho D_\rho + D_z \partial_\rho D_z = \frac{1}{\rho} \frac{\partial_z \rho}{(\partial_\psi \rho)^2} \partial_\psi \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) + \frac{1}{\rho (\partial_\psi \rho)^2} \partial_\psi \left(\frac{1}{\rho \partial_\psi \rho} \right)$$

$$\partial_\psi \left(\frac{f(D)}{D} \right) = \frac{1}{(2\pi)^2} \frac{1}{D^3} [D f'(D) - f(D)] \cdot \left[\frac{1}{\rho \partial_\psi \rho} \partial_\psi \left(\frac{1}{\rho \partial_\psi \rho} \right) + \frac{\partial_z \rho}{\rho \partial_\psi \rho} \partial_\psi \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) \right]$$

$$D \partial_z D = D_\rho \partial_z D_\rho + D_z \partial_z D_z$$

$$\partial_z \left(\frac{f(D)}{D} \right) = \frac{1}{(2\pi)^2} \frac{1}{D^3} [D f'(D) - f(D)] \left[\frac{\partial_z \rho}{\rho \partial_\psi \rho} \partial_z \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) + \frac{1}{\rho \partial_\psi \rho} \partial_z \left(\frac{1}{\rho \partial_\psi \rho} \right) \right]$$

$$D^2 = \frac{1}{(2\pi)^2 \rho^2} \left[\frac{1}{(\partial_\psi \rho)^2} + \frac{(\partial_z \rho)^2}{(\partial_\psi \rho)^2} \right]$$

$$* \quad D^3 \left[\frac{1}{\partial_\psi \rho} \partial_\psi \left(\frac{1}{\rho \partial_\psi \rho} \right) + \left(\frac{\partial_z \rho}{\partial_\psi \rho} \right) \partial_\psi \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) - \partial_z \left(\frac{\partial_z D}{\rho \partial_\psi \rho} \right) \right]$$

$$- \frac{1}{(2\pi)^2} \frac{g(D)}{\rho^2} \frac{1}{(\partial_\psi \rho)^2} \left[\partial_z \left(\frac{\partial_z \rho}{\rho \partial_\psi \rho} \right) - \partial_z \rho \, \partial_z \left(\frac{1}{\rho \partial_\psi \rho} \right) \right] = 0$$

$$\frac{1}{2\pi}\left[\quad\right] - \frac{g(D)\,\partial_{zz}\rho}{(2\pi\rho\,\partial_\psi\rho\,D)^3} = 0$$

$$2\pi\rho D = \pm\frac{\left[1+(\partial_z\rho)^2\right]^{\frac{1}{2}}}{\partial_\psi\rho}$$

$$-\ \frac{g(D)\,\partial_{zz}\rho\ \mathrm{sgn}(\partial_\psi\rho)}{\left[1+(\partial_z\rho)^2\right]^{3/2}} = 0$$

R很大时. D较小. 设 $f(D) = 1 + D^\sigma$ （$\sigma=1$ 即情况2）.

即 $g(D) = \sigma^{-1} D^{1-\sigma}$

这时 $|\partial_z\rho| \ll |\partial_\psi\rho|$，$\partial_\psi\rho < 0$，便有.

$$D \approx \frac{1}{2\pi\rho}\left(-\frac{1}{\partial_\psi\rho}\right)$$

$$g(D) \approx (-1)^\sigma\cdot\sigma\,(2\pi\rho)^{1-\sigma}\frac{1}{(\partial_\psi\rho)^{\sigma-1}}$$

$$\partial_{zz}\rho - \frac{\sigma(-1)^\sigma}{(2\pi)^\sigma}\cdot\frac{1}{(\partial_\psi\rho)^{\sigma-1}}\cdot\partial_\psi\left(\frac{1}{\rho\,\partial_\psi\rho}\right) = 0$$

即 $\partial_{zz}\rho - \frac{1}{(2\pi)^\sigma}\,\frac{1}{\partial_\psi\rho}\,\partial_\psi\left(-\frac{1}{\rho\,\partial_\psi\rho}\right)^\sigma = 0$ 可分离变量

$$\varepsilon = \rho^{-2} f^2$$

$$\ln\varepsilon = -2\ln\rho + 2\ln f$$

$$\ln\varepsilon + 2\ln\rho = 2\ln f$$

$$\nabla(\ln\varepsilon + 2\ln\rho) = \frac{2}{f}\nabla f$$

$$\nabla f = \frac{f}{2}\nabla(\ln\varepsilon + 2\ln\rho)$$ $\left(\frac{\partial\ln\varepsilon}{\partial\rho} + \frac{2}{\rho},\ \frac{\partial\ln\varepsilon}{\partial z}\right)$ $(\partial_\rho\Omega,\ \partial_z\Omega)$

$$\cancel{f}\,\nabla^2\Omega = 2\nabla f\cdot\nabla\Omega = \cancel{f}\,\nabla(\ln\varepsilon + 2\ln\rho)\cdot\nabla\Omega$$

$$\partial_{\rho\rho}\Omega + \partial_{zz}\Omega \mp \frac{1}{\rho}\partial_\rho\Omega = \frac{\partial\ln\varepsilon}{\partial\rho}\cdot\frac{\partial\Omega}{\partial\rho} + \frac{\partial\ln\varepsilon}{\partial z}\cdot\frac{\partial\Omega}{\partial z} \cancel{+\frac{2}{\rho}\partial_\rho\Omega}$$

§4 电动力学中的电荷屏蔽效应

经典电磁学中 介质的极化与介质中分子极化率有关.

量子电动力学中 光子(电磁波量子)可与正负电子对相互转化. 真空可视为特殊介质.

当一假想电荷置于真空中, 周围引起真空极化(真空中正负电子对与光子相互作用引起的).

极化场 + 原场 $\Rightarrow$ 为实际电场.

真空效应永远存在 裸电场 $\longrightarrow$ dressed.

由静电学 极化电荷与引起它的裸电荷符号相反 "屏蔽效应

是电磁学基本特征之一. 考虑到真空极化现象的电动力学仍适用.

△ 静电介质出发的简化模型 本应用量子电动力学 但此也可得出本质.

设带电粒子 裸电荷 e_0 置于介质中. 电荷密度为球对称分布 $\rho_0(r) = e_0 n_0(r)$ (4-1)

极化产生的感生电荷密度 $\rho_c(\vec{r}) = e_0 n_c(\vec{r})$ (4-2)

电场的势 u $\nabla^2 u(\vec{r}) = -4\pi e_0 (n_0(r) + n_c(\vec{r}))$ (4-3)

引入 ~~势~~ v 且 v ~~为~~ $e_0 u(\vec{r})$. 则 $v(\vec{r})$ 满足

$$\nabla^2 v(\vec{r}) = -4\pi e_0^2 (n_0(r) + n_c(\vec{r})) \quad (4-4)$$

一般, 复杂介质(如真空). 感生 ~~电荷~~ $n_c(\vec{r})$ 与 $v(\vec{r})$ 有关且全空间关联.

描述普遍关系 $n_c(\vec{r}) = \int d\vec{r}'\, \pi(\vec{r}, \vec{r}')\, v(\vec{r}')$

$\Downarrow$ 势 v 零点选择任意. 具平移不变性.

$\pi(\vec{r} - \vec{r}')$

若 $v(\vec{r})$ 与坐标无关. 为常数. 则无场强, 介质无极化. 要求.

$$n_c(\vec{r}) = const \cdot \int d\vec{r}'\, \pi(\vec{r} - \vec{r}') = 0.$$

$$\Rightarrow \int d\vec{r}'\, \pi(\vec{r} - \vec{r}') = 0. \quad (4-5)$$

弱场 电荷. 电场均不强, 极化也较弱. 可作弱场近似. 将 $n_c(\vec{r})$ 在 $\vec{r}' = \vec{r}$ 附近 Taylor展开

$$n_c(\vec{r}) = \int d\vec{\rho}\, \pi(\vec{\rho}) \left[v(\vec{r}) + (\nabla v)_{\vec{\rho}=0} \cdot \vec{\rho} + \frac{1}{2} \partial_i \partial_j v|_{\vec{\rho}=0} \cdot \rho_i \rho_j + \cdots \right]$$

$$\vec{\rho} \equiv \vec{r} - \vec{r}'$$

积分第一项 代入(4-5)式为 $\underline{0}$.

第二项 各向同性介质中 $\pi(\vec{\rho})$ 与 $\vec{\rho}$ 方向无关 仅 $\vec{\rho}$ 与方向有关.

? 对方位立体角的积分为 $\underline{0}$

第三项. 关系式. $\rho_i \rho_j = \frac{1}{3} \rho^2 \delta_{ij}$ 与 $\vec{r}$ 方向无关. $\Rightarrow n_c(\vec{r})$ 与 $\vec{r}$ 方向无关.

则 $n_c(r) = \frac{1}{6} \left(\int d\vec{\rho}\, \rho^2 \pi(\rho) \right) \nabla^2 v$ (4-6)

代入式(4-4)中. $v(\vec{r})$ 与 $\vec{r}$ 方向无关.

$$\nabla^2 v(r) = -4\pi e_0^2 n_0(r) - \frac{2}{3}\pi e_0^2 \left(\int \pi(\rho) \rho^2 d\vec{\rho} \right) \nabla^2 v.$$

$\nabla^2 v(r) = -4\pi e_0^2 n_0(r)$

*分析：$\varepsilon_0>1$ 介质极化 → 电荷屏蔽（√ 量子电动力学中预计真空介质）

$\varepsilon_0<1$ 反屏蔽

原则上计算积分 $\int \Pi(\rho)\rho^2 d\vec{\rho}$，特别是符号判定．严格计算超出静电学范畴．

用"量纲分析法"作简单讨论．

自然单位制中 $c=\hbar=1$．"[ ]"量纲．$[L]$ 长度量纲

$m=mc(=\hbar\tilde{\omega})=mc^2(\hbar\omega)=\tilde{\omega}=\omega$ $\quad [长度]=\frac{1}{[时间]}=[能量]=[质量]$

$[n_c]=[L]^{-3}$ $\quad [d\vec{r}']=[L]^3$ $\quad [v]=[L]^{-1}$

$\xRightarrow{代入\ n_c(\vec{r})=\int d\vec{r}'\,\Pi(\vec{r},\vec{r}')\,v(\vec{r}')}$ $\quad [\Pi(\rho)]=[L]^{-5}$

可设 $\Pi(\rho)=\frac{A}{\rho^5}$ $\quad A$ 为待定常数

$d\vec{\rho} \xrightarrow{球坐标下积分} \int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\,\rho^2\sin\theta$ $\quad -\cos\theta\Big|_0^{\pi}$

$$\int\Pi(\rho)\rho^2 d\vec{\rho}=4\pi A\int_{r_0}^{r}\frac{d\rho}{\rho}=4\pi A\ln\frac{r}{r_0}\overset{?}{=}2\pi A\ln\left(\frac{r}{r_0}\right)^2$$

式 (4-7) 变为
$$\nabla^2 v=\frac{-4\pi e_0^2 n_0(r)}{1+\frac{4}{3}\pi^2 e_0^2 A\ln\left(\frac{r}{r_0}\right)^2}$$
$$=\frac{-4\pi e_0^2(r)\,n_0(r)}{\varepsilon_0}\ \underline{\underline{记}}\ -4\pi\underbrace{e^2(r)}_{直}\,n_0(r)$$

$$e^2(r)=\frac{e_0^2}{1+\frac{4}{3}\pi^2 e_0^2 A\ln\left(\frac{r}{r_0}\right)^2}\ \underline{\underline{*注}}\ \frac{e_0^2}{1+\frac{e_0^2}{3\pi}A\ln\left(\frac{r}{r_0}\right)^2} \qquad (4-8)$$

*注：const A 任意．在此处被重新定义．

在量子电动力学对于真空的一阶微扰论中计算得 $A=1$．

$$\varepsilon_0=1+\frac{1}{3\pi}e_0^2\ln\left(\frac{r}{r_0}\right)^2$$

△根据 (4-8) 式 $\quad r=r_0 \quad e=e_0 \qquad r_0$ 为假想裸荷分布半径 ≈ 0．r 电子大致线度．

$r_c\sim\lambda_{e\,\text{Compton}}\sim\frac{1}{m_e}$

$\because r\gg r_0$ 故电动力学中 $\quad \varepsilon_0>1$，$\underset{\uparrow\ 实测电荷}{e^2(r_c)}<e_0^2$

※决定屏蔽的最重要因素在于电磁作用中 $A>0$．

△重整化

$e^2(r_c)\xrightarrow{索维霍德}$ 精细常数 α

(4-8) 式表明 $\alpha\sim r$ 距离 $\xrightarrow{测不准关系}$ 动量．利用 $\frac{r_c^2}{r_0^2}\approx\frac{k^2}{m_e^2}$

$e^2(r_c)=\alpha_c \qquad e_0^2=\alpha(k^2)$

$$\alpha_c=\frac{\alpha(k^2)}{1+\frac{1}{3\pi}\alpha(k^2)\ln\frac{k^2}{m_e^2}} \qquad (4-9)$$

$\downarrow$

当 $k^2\to\infty$ 时 (即 $r_0\to 0$), $\alpha(k^2)\to\alpha(\infty)$ 为裸荷. 无法直测.

(4-8), (4-9) 及 $A>0$ 均在微扰论下得到. 即 $\frac{\alpha(k^2)}{3\pi}\ln\frac{k^2}{m_e^2}\ll 1$

弱场近似, 量纲分析

将(4-9)式改写为 $\alpha(k^2)=\dfrac{\alpha_c}{1-\frac{\alpha_c}{3\pi}\ln\frac{k^2}{m_e^2}}$ (4-10)

说明: ① 当 α_c 为定值 ($=\frac{1}{137}$) 当 k^2 很大 ($r_0\to 0$ 电子近似为点) 使 $\frac{\alpha_c}{3\pi}\ln\frac{k^2}{m_e}\sim 1$ 时

$\alpha(k^2)|_{k\to\infty}\to\infty$

② 当裸荷为定值 作点近似 ($k^2\to\infty$) $\alpha_c\to 0$ 即电磁作用常数为 0.

朗道于1954年发现的"零荷定理".

$k^2\to\infty$ 微扰论不能使用. 启示量子电动力学的适用范围问题.

△渐近自由现象.

若存在一种相互作用使理论计算 $A<0$ (以 $A=-1$, $\varepsilon<1$ 为例).

$$\tilde{\alpha}(k^2)=\frac{\tilde{\alpha}_c}{1+\frac{\tilde{\alpha}_c}{3\pi}\ln\frac{k^2}{m_e^2}}\qquad \tilde{\alpha}_c=\frac{\tilde{\alpha}(k^2)}{1-\frac{\tilde{\alpha}(k^2)}{3\pi}\ln\frac{k^2}{m^2}}\qquad (4-11)$$

与 $A>0$ 时结论相反.

$\tilde{\alpha}_c>\tilde{\alpha}(k^2)$.

$\tilde{\alpha}_c$ 一定时. $k^2\to\infty$ $\quad \tilde{\alpha}(k^2)|_{k^2\to\infty}\to 0$. 即小距离时 作用荷→0 作用消失.

称为"渐近自由"。

常出现于非阿贝尔场理论, 量子色动力学属于该理论。

△ 光子 与胶子.

电磁理论 — 屏蔽效应.
相互作用是电磁波
(光子) "荷"相互作用.
不带电.
若带电将产生B-S力
同性荷同向流动产生吸引
若吸 > 斥 力的性质将改变

非阿贝尔理论 —— 反屏蔽效应,
胶子 (传播荷)
自身带有 → 色荷
⇓
反屏蔽和渐近自由.

超出微扰理论 讨论 电动力学 $k^2\to\infty$ 紫外发散. 很难...
色动力学 $k^2\to 0$. 红外发散.

真空极化的屏蔽或反屏蔽性质决定了作用荷的渐近行为.