

Numerical Recipes in Fortran 77

The Art of Scientific Computing
Second Edition

Volume 1 of
Fortran Numerical Recipes

William H. Press

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