Charleton University Mathematics Journal Vol 1

Contents

Charleton University Mathematics Journal
Volume **1**, Number **1**, October **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **1-10**

MONOTONICITY METHODS IN *PDE*

ZUJIN ZHANG

ABSTRACT. In this paper, we renormalize the huts 5.1.3 and 6.1.1 in [1], so as to be more accessible, see more details in [4]. Roughly speaking, monotonicity is the natural substitution of convexity in building solutions of *PDE* .

CONTENTS

1. Minty-Browder method in L^2.

In this hut, we introduce the **monotonicity method** due to Minty and Browder. As as illustrative problem, we consider the following quasi-linear *PDE* :

$$\begin{cases} -\mathrm{div}\,(\boldsymbol{E}(Du)) = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{1}$$

where $\boldsymbol{E} : \mathbb{R}^n \to \mathbb{R}^n$ is given.

Observe that (1) can be solved by calculations of variations in case $\boldsymbol{E} = DF$ for some convex $F : \mathbb{R}^n \to \mathbb{R}$.

Key words and phrases. Monotonicity method, weak convergence method.

Our problem is then what natural conditions on $\boldsymbol{E}$ so that (1) may be directly tackled, when $\boldsymbol{E}$ is no longer the gradient of a convex function.

This is the work of Minty and Browder, who give

Definition 1. *A vector field $\boldsymbol{E}$ on $\mathbb{R}^n$ is called* **monotone** *if*

$$(\boldsymbol{E}(p) - \boldsymbol{E}(q)) \cdot (p - q) \geq 0, \ \forall \ p, q \in \mathbb{R}^n, \tag{2}$$

and show (1) can be tacitly worked out as

Theorem 2. *Assume $\boldsymbol{E}$ is monotone and satisfies the growth condition $|\boldsymbol{E}(p)| \leq C\,(1 + |p|)$, $p \in \mathbb{R}^n$.*

Let $\{u_k\} \in H_0^1(U)$ be weak solutions of the approximating problems

$$\begin{cases} -div\ (\boldsymbol{E}(Du_k)) = f_k, & in\ U, \\ u_k = 0, & on\ \partial U, \end{cases} \tag{3}$$

with $f_k \to f$ in $L^2(U)$.

Suppose $u_k \rightharpoonup u$ in $H_0^1(U)$. Then u is a weak solution of (1).

Proof. We first write down

$$\begin{aligned} 0 &\leq \int_U [\boldsymbol{E}(Du_k) - \boldsymbol{E}(Dv)]\,[Du_k - Dv]\,dx \ \ (Monotonicity) \\ &= \int_U [f_k u_k - f_k v - \boldsymbol{E}(Dv)(Du_k - Dv)]\,dx, \ \forall \ v \in H_0^1(U) \\ &\quad \big(\text{integration by parts and weak formulation}\big). \end{aligned}$$

Then taking $k \to \infty$ yields

$$0 \leq \int_U [f(u - v) - \boldsymbol{E}(Dv) \cdot (Du - Dv)]\,dx.$$

Choosing $v = u + \lambda w$, with $\lambda \in \mathbb{R}$, $w \in H_0^1(U)$ furthermore gives

$$0 \leq sgn(\lambda) \int_U [\boldsymbol{E}(Du + \lambda Dw) \cdot Dw - fw]\,dx.$$

Passing $\lambda \to 0$ finally, we have as desired

$$0 = \int_U [\boldsymbol{E}(Du) \cdot Dv - fw]\, dx, \ \forall\, w \in H_0^1(U).$$

□

2. **Minty-Browder method in L^∞.** We consider the strong solutions of *PDE* , instead of weak solutions in (1). Hence the Minty-Browder method moves from L^2 to L^∞.

To illustrate how it works, let us consider the following fully non-linear *PDE* :

$$\begin{cases} F(D^2u) = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{4}$$

where $F : S^{n\times n} \to \mathbb{R}$ is given. Here $S^{n\times n}$ is the space of real, symmetric $n \times n$ matrices.

Definition 3. *The problem* (4) *is* **elliptic**, *if F is monotone decreasing with respect to matrix ordering on $S^{n\times n}$, and so*

$$F(S) \le F(R), \text{ if } S \ge R, \ S, R \in S^{n\times n}. \tag{5}$$

Remark 4. *This very definition of ellipticity, coincides with the classical ones. In fact, we say PDE*

$$Tr\,[A \cdot Du] = f$$

is **elliptic** *if A is a non-positive definite symmetric matrix. One then readily verifies*

$$\begin{aligned} S \ge R \quad &\Rightarrow \quad S - R \text{ non-negative definite} \\ &\Rightarrow \quad Tr\,[A \cdot (S - R)] \le 0 \\ &\Rightarrow \quad Tr[A \cdot S] \le Tr[A \cdot R], \ S, R \in S^{n\times n}. \end{aligned}$$

Now, suppose $f_k \to f$ uniformly, and consider the approximating problems

$$\begin{cases} F(Du_k) = f_k, & \text{in } U, \\ u_k = 0, & \text{on } \partial U. \end{cases} \tag{6}$$

Assume (6) has a smooth solution u_k, a priori bounded in $W^{2,\infty}(U)$. Then, up to a subsequence,

$$u_k \to u \text{ uniformly}, \; D^2u_k \overset{*}{\rightharpoonup} D^2u \text{ in } L^\infty(U; S^{n\times n}),$$

for some u.

Our **problem** is then: does u satisfies (4)?

If F is uniformly elliptic and convex, then strong estimates are available and passing to limit is simple, see [3]. The main interest is consequently for the nonconvex F, as in hut 1.

Recall that in hut 1, the main assumption leading to the existence of a weak solution is the monotonicity inequality (2). We shall then furnish a similar monotonicity in this current circumstance, replacing the ellipticity of F.

For this purpose, we need

Proposition 5. *Let $(X, \|\cdot\|)$ be a Banach space. Then the limit*

$$[f, g] = \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \tag{7}$$

exists for all $f, g \in X$.

Proof. Writing

$$\frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} = \frac{\|g+\lambda f\| + \|g\|}{2} \cdot \frac{\|g+\lambda f\| - \|g\|}{\lambda},$$

we need only show that $\left\{\frac{\|g+\lambda f\| - \|g\|}{\lambda}\right\}_{\lambda>0}$ is bounded from below and increasing in λ. In fact, we have

1. $\dfrac{\|g+\lambda f\|-\|g\|}{\lambda} \geq \dfrac{-\lambda\|f\|}{\lambda} = -\|f\|$;
2. for $0<\lambda<\tilde{\lambda}$,

$$\begin{aligned} & \frac{\|g+\lambda f\|-\|g\|}{\lambda} - \frac{\left\|g+\tilde{\lambda} f\right\|-\|g\|}{\tilde{\lambda}} \\ = \; & \frac{\left\|\tilde{\lambda} g+\lambda\tilde{\lambda} f\right\| - \tilde{\lambda}\|g\| - \left\|\lambda g+\lambda\tilde{\lambda} f\right\| + \lambda\|g\|}{\lambda\tilde{\lambda}} \\ \leq \; & \frac{\left\|(\tilde{\lambda}-\lambda) g\right\| - (\tilde{\lambda}-\lambda)\|g\|}{\lambda\tilde{\lambda}} = 0. \end{aligned}$$

□

Remark 6. *In case X is a Hilbert space, $[f,g]$ is simply the inner product.*

We now give an useful property of $[\cdot,\cdot]$ as

Proposition 7. *The map $X\times X \ni \{f,g\} \mapsto [f,g]$ is upper semicontinous, that is,*

$$\limsup_{n\to\infty}\,[f_n,g_n] \leq [f,g]\,, \tag{8}$$

for all $f,g\in X$, $f_n\to f$, $g_n\to g$ in X.

Proof. Observe that in the proof of (7), we have $\left\{\dfrac{\|g+\lambda f\|-\|f\|}{\lambda}\right\}_{\lambda>0}$ is increasing in λ, for $f,g\in X$ fixed.

Thus

$$\begin{aligned} \limsup_{n\to\infty}\,[f_n,g_n] &= \limsup_{n\to\infty}\lim_{\lambda\to 0_+} \frac{\|g_n+\lambda f_n\|^2-\|g_n\|^2}{2\lambda} \\ &= \limsup_{n\to\infty}\left\{\lim_{\lambda\to 0_+}\left[\frac{\|g_n+\lambda f_n\|+\|g_n\|}{2}\cdot\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda}\right]\right\} \\ &= \limsup_{n\to\infty}\left[\|g_n\|\cdot\lim_{\lambda\to 0_+}\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda}\right] \\ &\leq \|g\|\cdot\limsup_{n\to\infty}\frac{\|g_n+\lambda f_n\|-\|g_n\|}{\lambda} \end{aligned}$$

$$\leq \quad \|g\| \cdot \frac{\|g+\lambda f\| - \|g\|}{\lambda}, \ \forall \lambda > 0.$$

Taking $\lambda \to 0_+$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} [f_n, g_n] &= \|g\| \cdot \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\| - \|g\|}{\lambda} \\
&= \lim_{\lambda\to 0_+} \frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \\
&= [f, g].
\end{aligned}$$

□

Then an explicit formula in case $X = C(\bar{U})$ as

Proposition 8. *Let $X = C(\bar{U})$, then*

$$[f, g] = \max\left\{f(x_0)g(x_0);\ x_0 \in \bar{U}, |g(x_0)| = \|g\|_{C(\bar{U})}\right\},\ f, g \in C(\bar{U}). \tag{9}$$

Proof. Denote by

$$M_h = \left\{x \in \bar{U};\ |h(x)| = \|h\|\right\},\ h \in C(\bar{U}).$$

Then

1. due to

$$\frac{\|g+\lambda f\|^2 - \|g\|^2}{2\lambda} \geq \frac{(g(x_0) + \lambda f(x_0))^2 - g(x_0)^2}{2\lambda} = g(x_0)f(x_0),\ \forall\ x_0 \in M_g,$$

we have

$$[f, g] \geq RHS \text{ of } (9).$$

2. for any sequence $\{\lambda_n\} \searrow 0$, $x_n \in M_{g+\lambda_n f}$,

$$\begin{aligned}
\frac{\|g+\lambda_n f\|^2 - \|g\|^2}{2\lambda_n} &\leq \frac{(g(x_n) + \lambda_n f(x_n))^2 - g(x_n)^2}{2\lambda_n} \\
&= f(x_n)g(x_n) + \frac{\lambda_n}{2} f(x_n)^2 \\
&\to f(x_\infty)g(x_\infty),\ \text{as } n \to \infty,
\end{aligned} \tag{10}$$

for some $\bar{U} \ni x_\infty \leftarrow x_n$.

Meanwhile, taking $n \to \infty$ in

$$|g(x_n) + \lambda_n f(x_n)| = \|g + \lambda_n f\|\,,$$

gives

$$|g(x_\infty)| = \|g\|\,.$$

This together with (10) shows that

$$[f, g] \leq RHS \text{ of } (9).$$

The proof is then completed. □

With this explicit formula for $[f, g]$, we show that monotonicity is a consequence of ellipticity as

Proposition 9. *If F is convex, then the operator $A[u] \equiv F(D^2u)$ satisfies*

$$0 \leq [A[u] - A[v], u - v]\,,\ \forall\, u, v \in C_0^2(\bar{U}). \tag{11}$$

Here $C_0^2(\bar{U})$ is the subspace of $C^2(\bar{U})$, with vanishing boundary data.

Proof. Suppose $(u - v)\,(x_0) = \|u - v\|_{C(\bar{U})}$, $x_0 \in U$, then

$$\begin{aligned}
& D^2(u-v)(x_0) \leq 0 \\
\Rightarrow\quad & F(D^2u)(x_0) \geq F(D^2v)(x_0)\ \big(\text{by ellipticity}\big) \\
\Rightarrow\quad & [A[u] - A[v], u - v] = \big(F(D^2u) - F(D^2v)\big)\,(x_0)\cdot(u - v)\,(x_0) \geq 0,
\end{aligned}$$

by invoking (9).

The case $(v - u)\,(x_0) = \|u - v\|_{C(\bar{U})}$, $x_0 \in U$ is similarly treated. □

With all the above preparations above, we now state and prove our main result in this hut.

Theorem 10. *Consider problem (4) and its approximating problems (6). If $A[u] \equiv F(D^2u)$ satisfies the monotonicity inequality:*

$$0 \leq [A[u] - A[v], u - v]\,,\ \forall\, u, v \in C_0^2(\bar{U}). \tag{12}$$

Then u solves (4) a.e..

Proof. 1. For the approximating solution $\{u_k\}$, we have

$$\begin{aligned} 0 &\leq [A[u_k] - A[v], u_k - v] \\ &\leq [f_k - A[v], u_k - v],\ \forall\, v \in C_0^2(\bar{U}). \end{aligned}$$

Taking $k \to \infty$ upon a subsequence, we obtain by invoking (8) that

$$0 \leq [f - A[v], u - v]\,,\ \forall\, v \in C_0^2(\bar{U}). \tag{13}$$

2. Our strategy to prove u solves (4) is then to choose appropriate v in (13).

In fact, since $u \in W^{2,\infty}(U)$, Rademacher's theorem (see [2, 5]) implies then u is C^2 a.e.. Fix any $x_0 \in U$ where $D^2u(x_0)$ exists. We **handcraft** a C^2 function v having the form

$$v(x) \begin{cases} \begin{array}{l} = u(x_0) + Du(x_0)(x - x_0) \\ + \frac{1}{2}D^2u(x_0)(x - x_0, x - x_0) + \varepsilon\,|x - x_0|^2 - 1 \end{array}, & x \text{ near } x_0; \\ = 0, & x \in \partial U; \\ \in \left(u(x) - \frac{1}{2}, u(x) + \frac{1}{2}\right), & \text{otherwise}. \end{cases} \tag{14}$$

(The $\varepsilon > 0$ is chosen so that $u - v$ looks like a parabola for x near x_0.) Then $|u - v|$ attains its maximum over $\bar{U}$ only at x_0. But then (13) and (9) say $(f - A[u])\,(x_0) \geq 0$, that is,

$$f(x_0) \geq F\left(D^2u(x_0) + 2\varepsilon I\right).$$

Sending $\varepsilon \to 0_+$, we find

$$f(x_0) \geq F(D^2u(x_0)).$$

The opposite inequality follows by replacing $\varepsilon |x - x_0|^2 - 1$ by $-\varepsilon |x - x_0|^2 + 1$ in (13). Consequently, we have

$$F(D^2u(x_0)) = f(x_0), \ a.e. x_0 \in U.$$

□

Acknowledgements. Thanks are due to the discussion group of Professor Yin at Sun Yat-sen University, in particular Dr. Liu's lectures on the monotone property of $\left\{\frac{||g + \lambda f|| - ||g||}{\lambda}\right\}_{\lambda>0}$ in the proof of (7), setting forth the simple observation of the proof of (8) by the author through suffering two misleading applications of L' Hospital's law in calculus.

REFERENCES

[1] L.C. Evans, Weak convergence methods for nonlinear partial differential equations. CBMS Regional Conference Series in Mathematics, 74. American Mathematical Society, 1990.

[2] H. Federer, Geometric measure theory. Springer-Verlag New York Inc., New York, 1969.

[3] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[4] J.L. Lions, Quelques mthodes de ré solution des problmes aux limites non linaires. Gauthier-Villars, Paris, 1969.

[5] L. Simon, Lectures on geometric measure theory. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, P.R. China

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Website: http://www.sciencenet.cn/u/zjzhang/
Volume **1**, Number **2**, November **2010**
pp. **11-15**

MEAN-VALUE PROPERTY OF THE HEAT EQUATION

ZUJIN ZHANG

ABSTRACT. In this paper, we detailed the proof of the mean-value theorem for the heat equation, see [1] for example.

Let $U \subset \mathbb{R}^n$ be open and bounded, and $T > 0$. We give

Definition 1. *1. The* **parabolic cylinder** *is the* **parabolic interior** *of* $\bar{U} \times [0, T]$*:*

$$U_T \equiv U \times (0, T].$$

2. The **parabolic boundary** *of* U_T *is*

$$\Gamma_T \equiv \bar{U}_T - U_T,$$

which comprises the bottom and vertical sides of $U \times [0, T]$*, but not the top.*

In this parabolic cylinder U_T, we want to derive a kind of analogue to the mean-value property for harmonic function. For this purpose, we introduce

Definition 2. *The* **heat ball** $E(x, t; r)(r > 0)$ *at* $(x, t) \in \mathbb{R}^{n+1}$ *is*

$$E(x, t; r) = \left\{ (y, s) \in \mathbb{R}^{n+1};\ \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}.$$

Remark 3. *1. The heat ball is a region in space-time, the boundary of which is a level set of* $\Phi(x - y, t - s)$.

Key words and phrases. heat equation, mean-value property, fundamental solution.

2. *Written explicitly, we have*

$$\frac{1}{[4\pi(t-s)]^{n/2}}e^{-\frac{|x-y|^2}{4(t-s)}} = \Phi(x-y, t-s) \geq \frac{1}{r^n},$$

$$r^n e^{-\frac{|x-y|^2}{4(t-s)}} \geq [4\pi(t-s)]^{n/2}.$$

Applying the logarithmical function, we obtain

$$n \ln r - \frac{|x-y|^2}{4(t-s)} \geq \frac{n}{2} \ln [4\pi(t-s)],$$

$$|x-y|^2 \leq 2n(t-s) \ln \frac{r^2}{4\pi(t-s)}.$$

One then verifies easily that RHS of the above inequality equal 0 *if*

$$s = t - \frac{r^2}{4\pi} \text{ or } s = t.$$

This echoes the notion of heat ball, a region in space-time, with the scale in t *is twice that in* x*.*

3. *By the above calculations, we find that the function*

$$\psi \equiv -\frac{n}{2} \ln [4\pi(t-s)] - \frac{|x-y|^2}{4(t-s)} + n \ln r, \tag{1}$$

vanishes on $\partial E(x, t; r)$*, which is helpful in integration by parts formula, as we shall in later on. Notice also that*

$$\psi_y = -\frac{y}{2(t-s)}, \tag{2}$$

$$\psi_s = \frac{n}{2}\frac{s}{t-s} - \frac{|x-y|^2}{4(t-s)^2}. \tag{3}$$

Now, we state and prove our mean-value theorem for the heat equation as

Theorem 4. *(A mean-value property for the heat equation). Let* $u \in C_1^2(U_T)$ *solve the heat equation. Then*

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|y|^2}{s^2} dyds, \tag{4}$$

for each $E(x,t;r) \subset U_T$.

Proof. 1. An useful identity:

$$\iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4, \tag{5}$$

where $E(1) = E(0,0;1)$.

Indeed,

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2} dyds &= \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} ds \int_{|y|^2 \le -2ns \ln \frac{1}{-4\pi s}} |y|^2 \, dy \\
&= \int_{-\frac{1}{4\pi}}^{0} \frac{ds}{s} \int_0^{[-2ns \ln \frac{1}{-4\pi s}]^{1/2}} n\alpha(n) r^{n-1+2} dr \\
&= \frac{n\alpha(n)}{n+2} \int_{-\frac{1}{4\pi}}^{0} \frac{1}{(-s)^2} \left[2\pi (-s) \ln \frac{1}{4\pi (-s)} \right]^{\frac{n+2}{2}} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \int_0^{\frac{1}{4\pi}} s^{\frac{n-2}{2}} \left(\ln \frac{1}{4\pi s} \right)^{\frac{n+2}{2}} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \int_\infty^0 \left(\frac{1}{4\pi} e^{-s} \right)^{\frac{n-2}{2}} \cdot s^{\frac{n+2}{2}} \cdot \left(-\frac{1}{4\pi} e^{-s} \right) ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \cdot \frac{1}{(4\pi)^{n/2}} \int_0^\infty s^{\frac{n+4}{2}-1} e^{-\frac{n}{2}s} ds \\
&= \frac{n\alpha(n)(2n)^{\frac{n+2}{2}}}{n+2} \cdot \frac{1}{(4\pi)^{n/2}} \int_0^\infty \left(\frac{2}{n} \right)^{\frac{n+4}{2}} t^{\frac{n+4}{2}-1} e^{-t} dt \\
&= \frac{8}{(n+2)\pi^{n/2}} \cdot \Gamma\left(\frac{n}{2} + 2 \right) \\
&= \frac{8}{(n+2)\pi^{n/2}} \cdot \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1 \right)} \cdot \left(\frac{n}{2} + 1 \right) \Gamma\left(\frac{n}{2} + 1 \right) \\
&= 4.
\end{aligned}$$

2. We now prove (4). Without loss of generality, we may assume that $(x,t)=(0,0)$. Write $E(r)=E(0,0;r)$ and set

$$\begin{aligned}\varphi(r) &\equiv \frac{1}{r^n}\iint_{E(r)} u(y,s)\frac{|y|^2}{s^2}dyds\\ &= \iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}dyds.\end{aligned}$$

Then

$$\begin{aligned}\varphi'(r) &= \iint_{E(1)}\left[y\cdot D_yu\frac{|y|^2}{s^2}+2rD_su\frac{|y|^2}{s}\right]dyds\\ &= \frac{1}{r^{n+1}}\iint_{E(r)}\left[y\cdot D_yu\frac{|y|^2}{s^2}+2D_su\frac{|y|^2}{s}\right]dyds\\ &\equiv A+B.\end{aligned}$$

Next, we calculate B as

$$\begin{aligned}B &= \frac{1}{r^{n+1}}\iint_{E(r)} 2D_su\frac{|y|^2}{s}dyds\\ &= \frac{4}{r^{n+1}}\iint_{E(r)} D_suD_y\varphi\cdot ydyds\ \ ((2))\\ &= -\frac{4}{r^{n+1}}\int_{E(r)} y\cdot D_sD_yu\varphi dyds-\frac{4n}{r^{n+1}}\iint_{E(r)} D_su\varphi dyds\\ &\quad \left(\text{integration by part w.r.t. } y\right)\\ &= \frac{4}{r^{n+1}}\iint_{E(r)}\left\{y\cdot D_yu\left[-\frac{n}{2s}-\frac{|y|^2}{4s^2}\right]\right\}dyds\\ &\quad -\frac{4n}{r^{n+1}}\iint_{E(r)} D_su\varphi dyds\ \left(\text{integration by part w.r.t. } s \text{ and } (3)\right)\\ &= -A+\frac{4}{r^{n+1}}\iint_{E(r)}\left[-\frac{n}{2s}y\cdot D_yu-n\Delta_yu\varphi\right]dyds\ \left(D_su-\Delta_yu=0\right)\\ &= -A\left(\text{integration by part w.r.t. } y \text{ and } (2)\right).\end{aligned}$$

Hence,

$$\varphi(r) = \lim_{t\to 0_+}\varphi(t)=\lim_{t\to 0_+}\iint_{E(1)} u(ry,r^2s)\frac{|y|^2}{s^2}dyds$$

$$= \iint_{E(1)} u(0,0)\frac{|y|^2}{s^2}dyds = 4u(0,0).$$

The proof of the mean-value property of the heat equation is thus completed. □

Acknowledgements. The author would like to thank Dr. Zhang at Sun Yat-sen University, who are still on his road to Alexandrov geometry, and who showed me

```
\usepackage{palatino}
```

to display with the font here.

REFERENCES

[1] L.C. Evans, Weak convergence methods for nonlinear partial differential equations. CBMS Regional Conference Series in Mathematics, 74. American Mathematical Society, 1990.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **3**, November **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **16-17**

ON SOLUTION FORMULAE OF IBVP FOR THE HEAT EQUATION

ZUJIN ZHANG

ABSTRACT. In this paper, we give a solution formula of the initial/boundary-value problem for the heat equation via reflection method. This problem is 2.5.13 in [1].

Given a smooth $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, we have the solution formula

$$u(x,t) = \frac{x}{4\pi} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)ds$$

for the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0, & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u = 0, & \text{on } \mathbb{R}_+ \times \{t = 0\}, \\ u = g, & \text{on } \{x = 0\} \times [0,\infty). \end{cases}$$

Proof. Setting $v(x,t) \equiv u(x,t) - g(t)$, due to the fact that

$$v = 0, \quad \text{on } \{x = 0\} \times [0,\infty),$$

we may odd reflect v. Still denoting the resulting function by v yields

$$\begin{cases} v_t - v_{xx} = \begin{cases} g_t, & x < 0 \\ -g_t, & x > 0 \end{cases}, & \text{in } \mathbb{R} \times (0,\infty), \\ v = 0, & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Solution formula for the heat equation in one dimension then gives

$$u(x,t) - g(t)$$

Key words and phrases. heat equation, solution formula, reflection method.

$$
\begin{aligned}
&= v(x,t)\\
&= \int_0^t g_s(s)ds\left[\int_{-\infty}^0 \Phi(x-y,t-s)dy - \int_0^\infty \Phi(x-y,t-s)dy\right]\\
&= \int_0^t g_s(s)\left[-\int_{-x}^x \Phi(y,t-s)dy\right]ds\\
&= -\int_0^t g(s)\left[\int_{-x}^x \Phi_t(y,t-s)dy\right]ds - g(t)\lim_{s\to t_-}\int_{-x}^x \Phi(y,t-s)dy\\
&= -\int_0^t g(s)\left[\int_{-x}^x \Phi_{yy}(y,t-s)dy\right]ds - 2g(t)\lim_{s\to t_-}\int_0^x \frac{1}{[4\pi(t-s)]^{1/2}}e^{-\frac{|y|^2}{4(t-s)}}dy\\
&= -\int_0^t g(s)\left[\Phi_y(t,y-s) = \frac{1}{[4\pi(t-s)]^{1/2}}\cdot\frac{-2y}{4(t-s)}e^{-\frac{|y|^2}{4(t-s)}}\Bigg|_{-x}^x\right]ds\\
&\quad -\frac{2g(t)}{\pi^{1/2}}\lim_{s\to t_-}\int_0^{\frac{x}{[4(t-s)]^{1/2}}} e^{-z^2}dz\\
&= \frac{x}{(4\pi)^{1/2}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{|x|^2}{4(t-s)}}g(s)ds - g(t)\left(\int_0^\infty e^{-z^2}dz = \frac{\pi^{1/2}}{2}\right).
\end{aligned}
$$

□

Acknowledgements. The author would like to thank Professor L.C. Evans for showing him a rough integrating by parts formula.

REFERENCES

[1] L.C. Evans, Partial differential equations. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **4**, November **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **18-20**

EQUIPARTITION OF ENERGY

XUANJI JIA AND ZUJIN ZHANG

ABSTRACT. In this paper, we show the equipartition of energy for the 1D wave equation [2], and suggest a challenging open problem.

Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h, & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1}$$

Suppose g, h have compact support. The **kinetic energy** is

$$k(t) \equiv \frac{1}{2} \int_{\mathbb{R}} u_t^2 dx,$$

and the **potential energy** is

$$p(t) \equiv \frac{1}{2} \int_{\mathbb{R}} u_x^2 dx.$$

Prove

1. $k(t) + p(t)$ is constant in t;
2. $k(t) = p(t)$ for all large enough times t.

Proof. 1. Since

$$\begin{aligned} \frac{d}{dt}[k(t) + p(t)] &= \int_{-\infty}^{\infty} [u_t u_{tt} + u_x u_{xt}]\, dx = \int_{-\infty}^{\infty} [u_t u_{xx} + u_x u_{xt}]\, dx \\ &= \int_{-\infty}^{\infty} [u_t u_x]_x\, dx = 0, \end{aligned}$$

Key words and phrases. equipartition of energy, wave equation, d'Alembert's formula, Paley-Wiener theorem, Brownian motion.

we see

$$k(t) + p(t) = k(0) + p(0) = \frac{1}{2} \int_{-\infty}^{\infty} \left[g'^2 + h^2 \right] dx.$$

2. In view of **d'Alembert's formula**,

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy,$$

and thus

$$u_t(x,t) = \frac{g'(x+t) - g'(x-t)}{2} + \frac{h(x+t) + h(x-t)}{2},$$

$$u_x(x,t) = \frac{g'(x+t) + g'(x-t)}{2} + \frac{h(x+t) - h(x-t)}{2}.$$

Consequently,

$$\begin{aligned} u_t^2 - u_x^2 &= [u_t + u_x] \cdot [u_t - u_x] \\ &= [g'(x+t) + h(x+t)] \cdot [-g'(x-t) + h'(x-t)] \\ &= -g'(x+t)g'(x-t) + g'(x+t)h(x-t) \\ &\quad -h(x+t)g'(x-t) + h(x+t)h(x-t) \\ &= 0, \text{ for all large } t, \end{aligned}$$

the last equality holding since both g and h have compact support:

$$\text{supp } (g,h) \subset [a,b]$$
$$\Rightarrow \text{either } x+t \text{ or } x-t \text{ leaves away } [a,b], \ \forall\, t > \frac{b-a}{2}, \ x \in \mathbb{R}.$$

We obtain finally that

$$k(t) - p(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[u_t^2 - u_x^2 \right] dx = 0,$$

for all large t.

□

Remark 1. *1. This result can be extended to the wave equation in general odd space dimensions. However, it involves Fourier analysis, mainly the Paley-Wiener theorem* [1].

2. To the authors' best knowledge, this equipartition of energy was first introduced by Einstein in 1901*s. Since then, many mathematicians have been devoted to studying this problem.*

3. Just in May 2010*, some experiments established in Texas showed that equipartition of energy was valid for Brownian motion.* **This would give a challenging and interesting open problem whether we can give a mathematical proof of it**.

Acknowledgements. The authors would like to thank C. Chen at CUHK for sending them a copy of [1].

REFERENCES

[1] R.J. Duffin, *E*quipartion of energy in wave motion, J. Math. Anal. Appl., **32** (1970), 386-391.

[2] L.C. Evans, *P*artial differential equations, American Mathematical Society, 1998.

DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA 321004, P. R. CHINA

E-mail address: jiamath@gmail.com

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **5**, November **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **21-24**

THE CURVATURE OF A 2D CURVE

ZUJIN ZHANG

ABSTRACT. In this paper, we establish various curvature formulae for a two dimensional curve.

1. **Arclength parametrization.** Let $\boldsymbol{\alpha}: I \to \boldsymbol{R}^2$ with $|\dot{\boldsymbol{\alpha}}| = 1$. Then

$$\langle \dot{\boldsymbol{\alpha}}, \ddot{\boldsymbol{\alpha}} \rangle = 0.$$

We call

$$\kappa(s) = |\ddot{\boldsymbol{\alpha}}| \tag{1}$$

the **curvature** of $\boldsymbol{\alpha}$ at s.

2. **Parametrization.** Assume a curve $C \subset \boldsymbol{R}^2$ is given parametrically as

$$\boldsymbol{\alpha}(t) = (x(t), y(t)).$$

Then the tangent vector

$$\boldsymbol{t} = \frac{(\dot{x}, \dot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

And thus

$$\begin{aligned} \kappa \boldsymbol{n} &= \frac{d\boldsymbol{t}}{dt} \cdot \frac{dt}{ds} \\ &= \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \cdot \frac{(\ddot{x}, \ddot{y}) \sqrt{\dot{x}^2 + \dot{y}^2} - (\dot{x}, \dot{y}) \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}}{\dot{x}^2 + \dot{y}^2} \end{aligned}$$

Key words and phrases. curvature, curve, arc length, polar coordinate, level set.

$$
\begin{aligned}
&= \frac{(\ddot{x}, \ddot{y}) (\dot{x}^2 + \dot{y}^2) - (\dot{x}, \dot{y}) (\dot{x}\ddot{x} + \dot{y}\ddot{y})}{(\dot{x}^2 + \dot{y}^2)^2} \\
&= \frac{(\dot{y} (\dot{y}\ddot{x} - \dot{x}\ddot{y}), \dot{x} (\dot{x}\ddot{y} - \dot{y}\ddot{x}))}{(\dot{x}^2 + \dot{y}^2)^2},
\end{aligned}
$$

$$
\kappa = \frac{|\ddot{x}\dot{y} - \dot{x}\ddot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{2}
$$

3. **Polar coordinates.** Let the curve C be given in polar coordinates as

$$
\rho = \rho(\vartheta), \quad a \leq \vartheta \leq b.
$$

Then

$$
\boldsymbol{t} = \frac{1}{\sqrt{\rho^2 + \dot{\rho}^2}} (\dot{\rho} \cos\vartheta - \rho \sin\vartheta, \dot{\rho} \sin\vartheta + \rho \cos\vartheta),
$$

$$
\begin{aligned}
\kappa \boldsymbol{n} &= \frac{d\boldsymbol{t}}{d\vartheta} \cdot \frac{d\vartheta}{ds} \\
&= \frac{1}{(\rho^2 + \dot{\rho}^2)^{3/2}} \cdot \Big[(\ddot{\rho} \cos\vartheta - 2\dot{\rho} \sin\vartheta - \rho \cos\vartheta, \ddot{\rho} \sin\vartheta + 2\dot{\rho} \cos\vartheta - \rho \sin\vartheta) \cdot \sqrt{\rho^2 + \dot{\rho}^2} \\
&\quad + (\dot{\rho} \cos\vartheta - \rho \sin\vartheta, \dot{\rho} \sin\vartheta + \rho \cos\vartheta) \cdot \frac{\rho\dot{\rho} + \dot{\rho}\ddot{\rho}}{\sqrt{\rho^2 + \dot{\rho}^2}} \Big],
\end{aligned}
$$

$$
\begin{aligned}
\kappa &= \frac{\sqrt{[(\rho^2 + \dot{\rho}^2)(\ddot{\rho} - \rho) - (\rho\dot{\rho} + \dot{\rho}\ddot{\rho})\dot{\rho}]^2 + [(\rho^2 + \dot{\rho}^2) \cdot 2\dot{\rho} - (\rho\dot{\rho} + \dot{\rho}\ddot{\rho})\rho]^2}}{(\dot{\rho}^2 + \ddot{\rho}^2)^2} \\
&= \frac{\sqrt{\rho^2 (\rho\ddot{\rho} - 2\dot{\rho}^2 - \rho^2)^2 + \dot{\rho}^2 (\rho\ddot{\rho} - 2\dot{\rho}^2 - \rho^2)^2}}{(\dot{\rho}^2 + \ddot{\rho}^2)^2} \\
&= \frac{|\rho\ddot{\rho} - 2\dot{\rho}^2 - \rho^2|}{(\rho^2 + \dot{\rho}^2)^{3/2}}.
\end{aligned}
$$

4. **Level sets.** Suppose at last the curve C is given by the level set of a function $u : \boldsymbol{R}^2 \to \boldsymbol{R}$ as

$$u\left(x(s), y(s)\right) = C,$$

for some $C \in \boldsymbol{R}$, and s is the arc length.

Then

$$Du \cdot \dot{\boldsymbol{\alpha}} = 0, \quad Du // \ddot{\boldsymbol{\alpha}};$$

$$D^2 u\left(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}\right) + Du \cdot \ddot{\boldsymbol{\alpha}} = 0;$$

$$\begin{aligned}
\kappa^2 &= \frac{1}{|Du|^2} \left| D^2 u\left(\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\alpha}}\right)\right|^2 \\
&= \frac{1}{|Du|^2} \left| \Delta u - D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) \right|^2 \\
&\quad \text{(The trace is an invariant of a matrix)} \\
&= \frac{1}{|Du|^4} \left| |Du|^2 \Delta u - D^2 u \left(Du, Du\right) \right|^2;
\end{aligned}$$

$$\begin{aligned}
\kappa &= \frac{1}{|Du|^2} \left| |Du|^2 \Delta u - D^2 u \left(Du, Du\right) \right| \\
&= \frac{1}{|Du|^2} \left| \left(u_{x_1}^2 + u_{x_2}^2\right)\left(u_{x_1x_1} + u_{x_2x_2}\right) - \left(u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2}\right) \right| \\
&= \frac{1}{|Du|^2} \left| u_{x_2}^2 u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_1}^2 u_{x_2x_2} \right|;
\end{aligned}$$

$$\kappa = \left| \frac{\text{div } \nabla u \cdot |Du|^2 - D^2 u\left(Du, Du\right)}{|Du|^2} \right| = \left| \text{div } \left(\frac{Du}{|Du|} \right) \right|.$$

In conclusion,

$$\kappa = \left| \text{div } \left(\frac{Du}{|Du|} \right) \right| = \frac{1}{|Du|^2} \left| u_{x_2}^2 u_{x_1x_1} - 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_1}^2 u_{x_2x_2} \right|. \quad (3)$$

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **6**, November **2010** Website: http://www.sciencenet.cn/u/zjzhang/
pp. **25-27**

DIRICHLET PRINCIPLE OF HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. We define harmonic maps as the critical point of Dirichlet energy functional. This is [1, 1.1].

Let

1. (M^n, g) be a Riemanninan manifold with or without boundary;
2. (N^l, h) be a compact Riemanninan manifold without boundary (closed).

The we may define the **Dirichlet energy functional**

$$E(u) = \int_M e(u)\mathrm{d}v_g,$$

where $e(u)$ is the **Dirichlet energy density function**, the expression of which in local coordinates (U, x^α), (V, u^i) is

$$e(u) \equiv \frac{1}{2}|\nabla u|_g^2 = \frac{1}{2}g^{\alpha\beta}(x)h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}.$$

Definition 1. *A map $u \in C^2(M, N)$ is a* **harmonic map** *if it is a critical point of the Dirichlet energy functional E.*

Proposition 2. *A map $u \in C^2(M; N)$ is a* **harmonic map** *iff u satisfies*

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0,\ in\ M,\ 1 \le i \le l.$$

Here

Key words and phrases. harmonic map, Dirichlet principle.

1. Δ_g is the **Laplace-Beltrami operator** *on (M,g) given by*

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right);$$

2. and Γ^i_{jk} is the **Christoffel symbol** *of the metric h on N given by*

$$\Gamma^i_{jk} = \frac{1}{2}h^{il}\left(h_{li,k} + h_{kl,j} - h_{jk,l}\right).$$

Proof. 1. Let $U \subset M$ be any coordinate chart and $\varphi \in C^2_0(U;\boldsymbol{R}^l)$. Then we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0}\left[\frac{1}{2}\int_M g^{\alpha\beta}h_{ij}(u+t\varphi)\left(u^i_\alpha + t\varphi^i_\alpha\right)\left(u^j_\beta + t\varphi^j_\beta\right)\sqrt{g}\mathrm{d}x\right] \\ &= \frac{1}{2}\int_M g^{\alpha\beta}h_{ij,k}(u)\varphi_k u^i_\alpha u^j_\beta \sqrt{g}\mathrm{d}x + \int_M g^{\alpha\beta}h_{ij}(u)u^i_\alpha\varphi^j_\beta\sqrt{g}\mathrm{d}x. \end{aligned}$$

2. Direct computations show

$$\begin{aligned} \int_M \Delta_g u^i h_{ij}(u)\varphi^j \mathrm{d}v_g &= \int_M \frac{\partial}{\partial x^\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial u^i}{\partial x^\beta}\right)h_{ij}(u)\varphi^i\mathrm{d}x \\ &= -\int_M \sqrt{g}g^{\alpha\beta}u^i_\beta\left(h_{ij,k}(u)u^k_\alpha\varphi^j + h_{ij}(u)\varphi^j_\alpha\right)\mathrm{d}x \\ &= -\frac{1}{2}\int_M \sqrt{g}g^{\alpha\beta}u^i_\alpha u^j_\beta\left(h_{ik,j} + h_{kj,i} - h_{ij,k}\right)(u)\varphi^k\mathrm{d}x \\ &= -\int_M g^{\alpha\beta}\Gamma^l_{ij}(u)h_{lk}(u)u^i_\alpha u^j_\beta\varphi^k\mathrm{d}v_g. \end{aligned}$$

This implies that

$$\Delta u^i + g^{\alpha\beta}\Gamma^i_{kl}(u)u^k_\alpha u^l_\beta = 0,\ \forall\ 1 \le i \le l.$$

□

REFERENCES

[1] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume 1, Number 7, December 2010
Website: http://www.sciencenet.cn/u/zjzhang/
pp. 28-29

INSTRINSIC VIEW OF HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. We see harmonic maps in an intrinsic point of view. This is [3, 1.2].

Viewing

$$du = du^i \frac{\partial}{\partial u^i} = \frac{\partial u^i}{\partial x^\alpha} \frac{\partial}{\partial u^i} \otimes dx^\alpha,$$

we may write

$$e(u) = \frac{1}{2} g^{\alpha\beta} h_{ij}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} = \frac{1}{2} \langle du, du \rangle_{T^*M \otimes u^*TN};$$

and noticing

$$u^* h \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) = h \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right) = h_{ij}(u) u^i_\alpha u^j_\beta,$$

we shall further see

$$e(u) = \frac{1}{2} tr_g (u^* h).$$

Proposition 1. [1, 2] *$u \in C^2(M; N)$ is a harmonic map iff u satisfies*

$$\tau(u) \equiv tr_g (\nabla du) = 0, \ in\ M.$$

Here ∇ is the covariant derivative on $T^*M \otimes u^*TN$.

Remark 2. *Component-wise, $\tau(u) = 0$ is equivalent to*

$$\tau^k(u) = g^{\alpha\beta} \left[u^k_{\alpha\beta} - \left(\Gamma^M \right)^\gamma_{\alpha\beta} u^k_\gamma + \left(\Gamma^N \right)^k_{ij} (u) u^i_\alpha u^j_\beta \right] = 0, \ in\ M, \ 1 \leq k \leq l.$$

Key words and phrases. harmonic map, intrinsic geometry.

Indeed,

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^\beta}} du &= \nabla_{\frac{\partial}{\partial x^\beta}} \left(u^i_\alpha \frac{\partial}{\partial u^i} \otimes dx^\alpha \right) \\
&= u^i_{\alpha\beta} \frac{\partial}{\partial u^i} \otimes dx^\alpha + \left(\Gamma^N\right)^k_{ij} (u) u^j_\beta \frac{\partial}{\partial u^k} \otimes dx^\alpha - \left(\Gamma^M\right)^\alpha_{\beta\gamma} u^i_\alpha \frac{\partial}{\partial u^i} \otimes dx^\gamma \\
&= \left[u^i_{\alpha\beta} - \left(\Gamma^M\right)^\gamma_{\alpha\beta} u^i_\gamma + \left(\Gamma^N\right)^i_{kj} (u) u^j_\alpha \right] \frac{\partial}{\partial u^i} \otimes dx^\alpha;
\end{aligned}$$

$$tr_g (\nabla du) = g^{\alpha\beta} \left[u^i_{\alpha\beta} - \left(\Gamma^M\right)^\gamma_{\alpha\beta} u^i_\gamma + \left(\Gamma^N\right)^i_{kj} (u) u^j_\alpha \right] \frac{\partial}{\partial u^i}.$$

REFERENCES

[1] J. Eells, L. Lemaire, *A report on harmonic maps.* Bull. London Math. Soc. **10** (1978) 1-68.

[2] J. Eells, L. Lemaire, *Another report on harmonic maps.* Bull. London Math. Soc. **20** (1988) 385-524.

[3] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **8**, December **2010** Website: http://www.sciencenet.cn/u/zjzhang/
pp. **30-32**

EXTRINSIC VIEW OF HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. We see harmonic maps in an extrinsic point of view.
This is [1, 1.3].

By the isometric embedding theorem of Nash [2], we may assume

$$\left(N^l, h\right) \hookrightarrow \boldsymbol{R}^L,$$

for some $L \geq 1$. Then

$$C^2(M; N) = \left\{u = \left(u^1, \cdot, u^L\right) \in C^2(M; \boldsymbol{R}^L);\ u(M) \subset N\right\},$$

and

$$e(u) = \frac{1}{2} g^{\alpha\beta} u^i_\alpha u^i_\beta.$$

Since N is a closed submanifold of $\boldsymbol{R}^L$, we can construct the **nearest point projection map**

$$\Pi_N :\ N_\delta \to N$$

1. where

$$N_\delta = \left\{y \in \boldsymbol{R}^L;\ d(y, N) \equiv \inf_{z \in n} |y - z| < \delta\right\};$$

2. for $y \in N_\delta$, $\Pi_N(y) \in N$ is such that

$$|y - \Pi_N(y)| = d(y, N);$$

Key words and phrases. harmonic map, extrinsic geometry.

3. and Π_N is smooth, the gradient of which,

$$P(y) = \nabla \Pi_N(y): \ \boldsymbol{R}^L \to T_y N$$

is an orthogonal projection; the Hessian of which, induces the second fundamental form of $N \subset \boldsymbol{R}^L$:

$$\begin{array}{rcccccl} A(y) \equiv \nabla P(y): & T_y N & \times & T_y N & \to & & \left(T_y N\right)^{\perp} \\ & (v & , & w) & \mapsto & & \sum_{i=l+1}^{L} Hess \Pi_N(v,w) \nu_i(y), \end{array}$$

where $\{\nu_i(y)\}_{i=l+1}^{L}$ is a local orthonormal frame of the normal bundle $\left(T_y N\right)^{\perp}$.

Now, we have

Proposition 1. *$u \in C^2(M;N)$ is a* **harmonic map** *iff u satisfies*

$$\Delta_g u \perp T_u N.$$

Proof. For $\varphi \in C_0^2(M;\boldsymbol{R}^L)$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \int_M |\nabla \Pi_N (u + t\varphi)|^2 \, \mathrm{d}v_g \\ &= 2 \int_M \langle \nabla u, \nabla (P(u)\varphi) \rangle \, \mathrm{d}v_g \\ &= -2 \int_M \left\langle \Delta_g u, P(u)\varphi \right\rangle \mathrm{d}v_g \\ &= -2 \int_M \left\langle P(u)\left(\Delta_g u\right), \varphi \right\rangle \mathrm{d}v_g. \end{aligned}$$

□

Remark 2. *Notice that*

$$\Delta_g u \perp T_u N$$

is equivalent to the PDE :

$$\Delta_g u + A(u)(\nabla u, \nabla u) = 0, \ \textit{in } M.$$

In fact, we may write

$$\Delta_g u = \sum_{i=l+1}^{L} \lambda_i(x)\nu_i(u),$$

with

$$\begin{aligned}\lambda_i &= \left\langle \Delta_g u, \nu_i(u)\right\rangle \\ &= div_{\,g}\left(\nabla u \cdot \nu_i(u)\right) - \nabla u \cdot \nabla\left(\nu_i(u)\right) \\ &= -\left(\nabla \nu_i\right)(u)\left(\nabla u, \nabla u\right) \\ &= -A(u)\left(\nabla u, \nabla u\right).\end{aligned}$$

Example 3. *Let*

(a) $M = T^n$ *be the n-dimensional flat torus;*

(b) and $N = S^k \subset \boldsymbol{R}^{k+1}$ *be the unit sphere.*

Then $u \in C^2(T^n, S^k)$ *is a harmonic map iff*

$$\Delta u + |\nabla u|^2 u = 0, \text{ in } T^n.$$

REFERENCES

[1] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

[2] J. Nash, *The imbedding problem for Riemannian manifolds.* Ann. of Math. **63** (1956) 20-63.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal Website: http://www.sciencenet.cn/u/zjzhang/
Volume **1**, Number **9**, December **2010** pp. **33-35**

A FEW FACTS ABOUT HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. We state some basic facts about the harmonic maps. This is [1, 1.5].

Proposition 1. *Let*

1. $\Phi : M \to M$ *a* C^2*-diffeomorphism;*
2. *and* $u \in C^2(M; N)$ *is a harmonic map with respect to* (M, g)*.*

Then

$$u \circ \Phi \in C^2(M; N) \text{ is a harmonic map with respect to } (M, \Phi^* g).$$

Proof. For $v \in C^2(M; N)$, we have

$$\frac{1}{2}\int_M |\nabla v|_g^2 \, \mathrm{d}v_g = \frac{1}{2}\int_M |\nabla (v \circ \Phi)|_{\Phi^* g}^2 \, \mathrm{d}v_{\Phi^* g}.$$

□

Proposition 2. *Let*

1. (M, g_1) *be a Riemann surface;*
2. $\Phi : (M, g_1) \to (M, g_2)$ *be a conformal map;*
3. *and* $u \in C^2(M; N)$ *is a harmonic map with respect to* (M, g_2)*.*

Then

$$u \circ \Phi \in C^2(M; N) \text{ is a harmonic map with respect to } (M, g_1).$$

Key words and phrases. harmonic map, diffeomorphism, conformal geometry.

Proof. By setting $\Phi^* g_2 = e^{2\varphi} g_1$, we have

$$\begin{aligned} E(v \circ \Phi, g_1) &= \frac{1}{2}\int_M tr_{g_1}\left((v\circ\Phi)^* h\right) \mathrm{d}v_{g_1} \\ &= \frac{1}{2}\int_M tr_{e^{-2\varphi}\Phi^* g_2}\left(\Phi^*\left(v^* h\right)\right) e^{-2\varphi} \mathrm{d}v_{\Phi^* g_2} \quad (n = dim\ M = 2) \\ &= \frac{1}{2}\int_M tr_{\Phi^* g_2}\left(\Phi^*\left(v^* h\right)\right) \mathrm{d}v_{\Phi^* g_2} \quad \left((cA)^{-1} = \frac{1}{c}A^{-1}\right) \\ &= \frac{1}{2}\int_M tr_{g_2}\left(v^* h\right) \mathrm{d}v_{g_2} \\ &= E(v, g_2), \end{aligned}$$

for all $v \in C^2(M; N)$. □

Remark 3. *1. Harmonic maps from S^1 to N correspond to* **closed geodesic** *in N.*

2. The set of harmonic maps from a Riemannian surface M depends only on the conformal structure of M.

3. Let $Id : (M, g) \to (M, g)$ be the identity map. then Id is a harmonic map.

Proof. Since $u(x) = Id(x) = x$, we have

$$\begin{aligned} \tau^k(u) &= g^{\alpha\beta}\left[u^k_{\alpha\beta} - \left(\Gamma^M\right)^\gamma_{\alpha\beta} u^k_\gamma + \left(\Gamma^N\right)^k_{ij}(u) u^i_\alpha u^j_\beta\right] \\ &= g^{\alpha\beta}\left[0 - \left(\Gamma^M\right)^\gamma_{\alpha\beta} \delta^k_\gamma + \left(\Gamma^M\right)^k_{ij} \delta^i_\alpha \delta^j_\beta\right] \\ &= 0. \end{aligned}$$

□

4. For $n = dim\ M = 2$, any conformal map $\Phi : (M, g_1) \to (M, g_2)$ is a harmonic map.

Proof.

$$(M, g_1) \xrightarrow{\Phi} (M, g_2) \xrightarrow{Id} (M, g_2).$$

□

REFERENCES

[1] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: `uia.china@gmail.com`

Charleton University Mathematics Journal Website: http://www.sciencenet.cn/u/zjzhang/
Volume **1**, Number **10**, December **2010** pp. **36-38**

BOCHNER IDENTITY FOR HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. Considered in this paper is one of the most important formulas for a harmonic map.

Theorem 1. *If $u \in C^2(M;N)$ is a harmonic map, then in a local coordinate system, there holds*

$$\Delta_g e(u) = |\nabla du|^2 + R^M_{\alpha\beta} u_\alpha u_\beta - R^N_{ijkl}(u) u^i_\alpha u^j_\beta u^k_\alpha u^l_\beta.$$

Proof. Fix an $x_0 \in M$, let (x_α) be a normal coordinate system around x_0, then

$$\begin{aligned} \Delta_g e(u) &= \partial_\beta \left(u_\alpha, u_{\beta\alpha}\right) \\ &= \left|u_{\alpha\beta}\right|^2 + \left\langle u_\alpha, u_{\beta\alpha,\beta}\right\rangle \\ &= \left|u_{\alpha\beta}\right|^2 + \left\langle u_\alpha, R^M_{\alpha\beta} u_\beta + u_{\beta\beta,\alpha}\right\rangle \\ &= \left|u_{\alpha\beta}\right|^2 + R_{\alpha\beta} u_\alpha u_\beta + \left\langle u_\alpha \left(\Delta_g u\right)_\alpha\right\rangle; \end{aligned}$$

$$\begin{aligned} \left|u_{\alpha\beta}\right|^2 &= \left|P(u)(u_{\alpha\beta})\right|^2 + \left|A(u)\left(u_\alpha, u_\beta\right)\right|^2 \\ &= |\nabla du|^2 + \left|A(u)(u_\alpha, u_\beta)\right|^2; \end{aligned}$$

$$\begin{aligned} \left\langle u_\alpha, \left(\Delta_g u\right)_\alpha\right\rangle &= -\left\langle u_\alpha, \left(A(u)\left(\nabla u, \nabla u\right)\right)_\alpha\right\rangle \\ &= \left\langle \Delta_g u, A(u)\left(\nabla u, \nabla u\right)\right\rangle \\ &= -\left\langle A(u)(\nabla u, \nabla u), A(u)(\nabla u, \nabla u)\right\rangle \end{aligned}$$

Key words and phrases. harmonic map, Bochner identity.

$$= \quad -\left\langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta)\right\rangle .$$

Thus

$$\Delta_g e(u) = |\nabla du|^2 + R^M_{\alpha\beta} u_\alpha u_\beta - R^N_{ijkl}(u) u^i_\alpha u^j_\beta u^k_\alpha u^l_\beta,$$

by Gauss-Kodazzi equations. □

Proposition 2. *Let*

1. (M, g) *be a closed manifold with* $Ric^M \geq 0$;
2. *the sectional curvature of* N, $K^N \leq 0$.

Then

1. *any harmonic map* $u \in C^2(M; N)$ *is totally geodesic.*
2. *If* $Ric^M > 0$ *at some point in* M, *then* u *is constant.*
3. *If* $K^N < 0$, *then either* u *is constant or* $u(M)$ *lies in a closed geodesic.*

Proof. 1.

$$\begin{aligned} & \Delta_g e(u) \geq 0 \\ \Rightarrow \quad & e(u) \text{ is subharmonic in } M \\ \Rightarrow \quad & e(u) \text{ is constant } \big(\text{maximum principle}\big). \end{aligned}$$

2.

$$\begin{aligned} & Ric^M(x_0) > 0 \\ \Rightarrow \quad & \nabla u(x_0) = 0 \\ \Rightarrow \quad & e(u) \equiv 0 \\ \Rightarrow \quad & u \text{ is constant.} \end{aligned}$$

3.

$$K^N < 0$$

$$\Rightarrow \quad \text{the linear span of } \left\{u^1, \cdots, u^n\right\} \text{ is at most of one dimension}$$

$$\Rightarrow \quad u(M) \begin{cases} \text{is a point} \\ \text{or lies in a closed geodesic} \end{cases} \text{in } N.$$

□

REFERENCES

[1] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **11**, December **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **39-42**

SECOND VARIATION FORMULA OF HARMONIC MAPS

ZUJIN ZHANG

ABSTRACT. The second variation formulae of harmonic maps into spheres and general target manifolds are derived. This is [1, 1.6].

Considered in this paper is the second variation formulae for harmonic maps into spheres and general target manifolds.

Proposition 1. *Let*

1. $u \in C^2(M; S^k)$ *is a harmonic map;*
2. $\varphi \in C_0^2(M; \boldsymbol{R}^{k+1})$.

Then

$$\frac{d^2}{dt^2}\Big|_{t=0}\left[\frac{1}{2}\int_M \left|\nabla\left(\frac{u+t\varphi}{|u+t\varphi|}\right)\right|^2 \mathrm{d}v_g\right] = \int_M \left(|\nabla\varphi|^2 - |\nabla u|^2\,|\hat{\varphi}|^2\right)\mathrm{d}v_g,$$

where $\hat{\varphi} = \varphi - \langle\varphi, u\rangle\, u$ *is the tangent component of* φ.

Proof. 1. For $\varphi \in C_0^2(M; \boldsymbol{R}^{k+1})$ and small $t \in \boldsymbol{R}$, by denoting

$$u_t = \frac{u+t\varphi}{|u+t\varphi|},$$

we have

$$\begin{aligned}\frac{du_t}{dt} &= \frac{\varphi\,|u+t\varphi| - (u+t\varphi)\,\frac{\langle u+t\varphi,\varphi\rangle}{|u+t\varphi|}}{|u+t\varphi|^2}\\ &= \frac{\varphi\,|u+t\varphi|^2 - (u+t\varphi)\,\langle u+t\varphi,\varphi\rangle}{|u+t\varphi|^3},\end{aligned}$$

$$\frac{du_t}{dt}\Big|_{t=0} = \varphi - \langle u,\varphi\rangle\,\varphi = \hat{\varphi};$$

Key words and phrases. harmonic map, second variation formula.

$$\begin{aligned}\frac{d^2u_t}{dt^2} &= \frac{1}{|u+t\varphi|^3}\left[2\varphi\langle u+t\varphi,\varphi\rangle - \varphi\langle u+t\varphi,\varphi\rangle - (u+t\varphi)|\varphi|^2\right] \\ &\quad \left[\varphi|u+t\varphi|^2 - (u+t\varphi)\langle u+t\varphi,\varphi\rangle\right]\frac{-3/2}{\left[|u+t\varphi|^2\right]^{5/2}}2\langle u+t\varphi,\varphi\rangle,\end{aligned}$$

$$\begin{aligned}\frac{d^2u_t}{dt^2}|_{t=0} &= 2\varphi\langle u,\varphi\rangle - \varphi\langle u,\varphi\rangle - u|\varphi|^2 - 3\langle u,\varphi\rangle[\varphi - u\langle u,\varphi\rangle] \\ &= 3\langle u,\varphi\rangle^2 u - |\varphi|^2 u - 2\langle u,\varphi\rangle\varphi.\end{aligned}$$

2. Direct computations show

$$\begin{aligned}&\frac{d^2}{dt^2}|_{t=0}\left[\frac{1}{2}\int_M\left|\nabla\frac{u+t\varphi}{|u+t\varphi|}\right|^2 \mathrm{d}v_g\right] \\ =& \int_M\left[\left|\nabla\left(\frac{du_t}{dt}|_{t=0}\right)\right|^2 + \left\langle\nabla u,\nabla\left(\frac{d^2u_t}{dt^2}|_{t=0}\right)\right\rangle\right]\mathrm{d}v_g \\ =& \int_M\left[|\nabla\hat{\varphi}|^2 - \left\langle\Delta_g u, 3\langle u,\varphi\rangle^2 u - |\varphi|^2 u - 2\langle u,\varphi\rangle\varphi\right\rangle\right]\mathrm{d}v_g \\ =& \int_M\left[|\nabla\hat{\varphi}|^2 + |\nabla u|^2\left(3\langle u,\varphi\rangle^2 - |\varphi|^2 - 2\langle u,\varphi\rangle^2\right)\right]\mathrm{d}v_g \\ =& \int_M\left[|\nabla\hat{\varphi}|^2 - |\nabla u|^2\left(|\varphi|^2 - \langle u,\varphi\rangle^2\right)\right]\mathrm{d}v_g \\ =& \int_M\left[|\nabla\hat{\varphi}|^2 - |\nabla u|^2|\hat{\varphi}|^2\right]\mathrm{d}v_g.\end{aligned}$$

□

Proposition 2. *Let*

1. *$u \in C^2(M;N)$ be a harmonic map;*
2. *$u_t \in C^2([0,1]\times M;N)$ be a family of* **smooth variations** *of u, i.e. $u_0 = u$.*

Then

$$\frac{d^2}{dt^2}|_{t=0}\left[\frac{1}{2}\int_M|\nabla u_t|_g^2\,\mathrm{d}v_g\right] = \int_M\left[|\nabla v|_g^2 - tr_g\left\langle R^N(v,\nabla u)v,\nabla u\right\rangle\right]\mathrm{d}v_g,$$

where

$$v = \frac{du_t}{dt}|_{t=0} \in C^2(M; u^*TN).$$

In particular,

$$K^N \leq 0 \Rightarrow u \text{ is stable: } \frac{d^2}{dt^2}|_{t=0}\left[\frac{1}{2}\int_M |\nabla u_t|_g^2 \, \mathrm{d}v_g\right] \geq 0.$$

Proof. 1. In local coordinates,

$$\frac{d}{dt}|_{t=0}\frac{\partial u_t}{\partial x^\alpha} = \nabla^{u^*TN}_{\frac{\partial}{\partial t}}\frac{\partial u_t}{\partial x^\alpha}|_{t=0} = \nabla^{u^*TN}_{\frac{\partial}{\partial x^\alpha}} v;$$

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\frac{\partial u_t}{\partial x^\alpha} &= \nabla^{u^*TN}_{\frac{\partial}{\partial t}}\nabla^{u^*TN}_{\frac{\partial}{\partial x^\alpha}}\frac{\partial u_t}{\partial t}|_{t=0} \\ &= \nabla^{u^*TN}_{\frac{\partial}{\partial x^\alpha}}\nabla^{u^*TN}_{v} v + R^N\left(\frac{\partial u}{\partial x^\alpha}, v\right) v.\end{aligned}$$

2. Direct computations show

$$\begin{aligned}&\frac{d^2}{dt^2}|_{t=0}\left[\frac{1}{2}\int_M |\nabla u_t|_g^2 \, \mathrm{d}v_g\right] \\ = \; &\int_M \left[|\nabla v|_g^2 + \left\langle \nabla u, \nabla\left(\frac{d^2 u_t}{dt^2}|_{t=0}\right)\right\rangle\right]\mathrm{d}v_g \\ = \; &\int_M \left[|\nabla v|_g^2 + \left\langle \frac{\partial u}{\partial x^\alpha}, \nabla^{u^*TN}_{\frac{\partial}{\partial x^\alpha}}\nabla^{u^*TN}_{v} v\right\rangle + tr_g\left\langle \boldsymbol{R}^N(\nabla u, v)\, v, \nabla u\right\rangle\right]\mathrm{d}v_g \\ = \; &\int_M \left[|\nabla v|_g^2 - \left\langle \nabla^{u^*TN}_{\frac{\partial}{\partial x^\alpha}}\frac{\partial u}{\partial x^\alpha}, \nabla^{u^*TN}_{v} v\right\rangle_g - tr_g\left\langle R^N(v, \nabla u)v, \nabla u\right\rangle\right]\mathrm{d}v_g \\ = \; &\int_M \left[|\nabla v|_g^2 - tr_g\left\langle R^N(v, \nabla u)\, v, \nabla u\right\rangle\right]\mathrm{d}v_g.\end{aligned}$$

□

REFERENCES

[1] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd. 2008.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **12**, December **2010**
Website: http://www.sciencenet.cn/u/zjzhang/
pp. **43-48**

AN EXISTENCE THEOREM FOR STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

ZUJIN ZHANG

ABSTRACT. In this paper, we show the existence of a solution to the stationary compressible Navier-Stokes equations under Dirichlet boundary conditions. This is [1, Page 121], and is delivered on Dec. 4th, 2010.

Theorem 1. *(Existence/Dirichlet BVP). Let* $\gamma = 5/3$, $N = 3$, $p \in (1,2)$. *Then* $\exists$ *a continuum* $C\left(\subset L^q \cap W^{1,q}, 1 \leq q < 2\right)$ *of solutions of*

$$\begin{cases} div\ (\rho u) = 0, \\ div\ (\rho u \otimes u) - \mu\Delta u - \xi\nabla div\ u + a\nabla\rho^\gamma = \rho f + g, \end{cases} \quad in\ \Omega \tag{1}$$

such that

1. $C \cap \{(\rho, u, M)\,;\ 0 \leq M \leq R\}$ *is bounded in* $L^2 \times H^1_0$, $\forall\, R > 0$;
2. $(0, u_0) \in C$ *where* u_0 *satisfies*

$$\begin{cases} -\mu\Delta u_0 - \xi\nabla div\ u_0 = g, & in\ \Omega, \\ u_0 = 0, & on\ \partial\Omega; \end{cases} \tag{2}$$

3. $\forall\, M > 0$, $\exists\ (\rho, u) \in C$ *such that* $\int_\Omega \rho^p = M$.

Proof. **Step I: Bounds for solution of the approximate problems:**

$$\begin{cases} \alpha\rho^p + \mathbf{div}\ (\rho u) = \alpha\frac{M}{|\Omega|}, \\ \alpha\rho^p u + \mathbf{div}\ (\rho u \otimes u) - \mu\Delta u - \xi\nabla\mathbf{div}\ u + a\nabla\rho^\gamma = \rho f + g, \end{cases} \quad \mathbf{in}\ \Omega. \tag{3}$$

1. $\int_\Omega \rho^p = M$;

Key words and phrases. existence of a solution, compressible Navier-Stokes equations, Dirichlet boundary value problem.

2. $\|u\|_{H^1} \leq C\left(1 + \|\rho\|_{6/5}\right)$, $\|\rho\|_\gamma \leq C\left(1 + \|u\|_{H^1}^{3/2}\right)$; which follows form the energy identity:

$$\int_\Omega \left\{ \alpha \frac{M}{|\Omega|}\frac{|u|^2}{2} + \alpha\rho^p \frac{|u|^2}{2} + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^\gamma - h\rho^{\gamma-1}\right) \right.$$
$$\left. + \mu|Du|^2 + \xi|\text{div } u|^2 - \rho u\cdot f - u\cdot g \right\} = 0.$$

3. $\|\rho\|_2 \leq C$, $\|u\|_{H^1} \leq C$.

Direct computations show

$$\begin{aligned}
\|\rho^\gamma\|_r &\leq \left\|\rho^\gamma - \fint_\Omega \rho^\gamma\right\|_r + |\Omega|^{1/r}\fint_\Omega \rho^\gamma \\
&\leq C\|\nabla\rho^\gamma\|_{W^{-1,r}} + C + C\|u\|_{H^1}^{5/2} \\
&\leq C + C\left\|\rho|u|^2\right\|_r + C\|u\|_{H^1}^{5/2} \\
&\leq C + C\|\rho\|_{\gamma r}\left\||u|^2\right\|_{\frac{\gamma}{\gamma-1}r} + C\|u\|_{H^1}^{5/2} \\
&\leq C + C\|\rho\|_{\gamma r}\|u\|_6^2 + C\|u\|_{H^1}^{5/2} \ \ (\text{if } \gamma r = 3(\gamma-1) = 2).
\end{aligned}$$

Thus

$$\|\rho\|_2^\gamma \leq C\left(1 + \|\rho\|_2\|\nabla u\|_2^2 + \|u\|_{H^1}^{5/2}\right),$$

$$\|\rho\|_2^{1/3} \leq C\left(1 + \|\rho\|_{6/5}\right).$$

To proceed further, we split into two cases.

(a) When $6/5 \leq p < 2$, $\|\rho\|_{6/5} \leq |\Omega|^{1/p-5/6}\|\rho\|_p \leq C$.

(b) In case $1 < p < 6/5$, $\|\rho\|_{6/5} \leq \|\rho\|_p^{1-\vartheta}\|\rho\|_2^\vartheta$ with

$$\frac{5}{6} = \frac{1-\vartheta}{p} + \frac{\vartheta}{2} \Rightarrow \vartheta = \frac{6-5p}{3(2-p)} < \frac{1}{3}.$$

Step II: The second approximation scheme and continuum.

We approximate (3) further by

$$\left.\begin{array}{l}
\alpha\rho^p + \text{div }(\rho u) - \varepsilon\Delta\rho = \frac{\alpha M}{|\Omega|}, \\
\frac{\alpha M}{|\Omega|}\frac{u}{2} + \frac{1}{2}\rho u\cdot\nabla u + \alpha\rho^p\frac{u}{2} + \frac{1}{2}\text{div }(\rho u\otimes u) \\
\quad -\mu\Delta u - \xi\nabla\text{div } u + a\nabla\rho^\gamma + \delta\nabla\rho^2 = \rho f + g,
\end{array}\right\} \text{ in } \Omega,$$
$$\frac{\partial\rho}{\partial n} = 0, \quad u = 0, \qquad \text{on } \partial\Omega, \tag{4}$$

where $\varepsilon, \delta \in (0,1]$. Here we add **viscosity and artificial pressure**.

We shall next establish the existence of a continuum (parameterized by M) of solutions of (4), and by taking $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, in the next step, to conclude the proof of Theorem 1.

Before invoking Leray-Schauder's fixed point theorem to show such a solution continuum, we first establish some a priori estimates, which shall be useful later on.

1. $\int_\Omega \rho^p = M$.
2. Energy identity:

$$\int_\Omega \left\{ \frac{\alpha}{2} h |u|^2 + \frac{1}{2}\alpha\rho^p |u|^2 + \mu |Du|^2 + \xi |\text{div } u|^2 + \varepsilon a\gamma\rho^{\gamma-2} |\nabla\rho|^2 + 2\varepsilon\delta |\nabla\rho|^2 \right.$$
$$\left. \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} - h\rho^{\gamma-1}\right) + 2\delta\alpha\left(\rho^{p+1} - h\rho\right) \right\} = \int_\Omega \{\rho u \cdot f + u \cdot g\}.$$

3. $\|\rho\|_3 \le C$, $\|u\|_{H^1} \le C$, independent of $\varepsilon \in (0,1]$.

   Notice that the improved regularity of ρ comes from the artificial pressure:

$$\frac{5}{3} \to 2, \quad 2 \to 3.$$

We now show the existence of a solution continuum $C_\alpha^{\delta,\varepsilon}$ to (4) by invoking the following

Theorem 2. *(Leray-Schauder). Let X be a Banach space, and $T : X \times [0,1] \to X$ be compact. Assume*

1. *$T(x,0) = x$, $\forall\, x \in X$;*
2. *$\exists\, M > 0$, s.t. $x = T(x,\sigma)$, $\sigma \in [0,1] \Rightarrow \|x\| \le M$.*

Then $T(\cdot,1)$ has a fixed point.

The Banach space we live is chosen to be $X = W^{1,\infty} \times \left(W^{1,\infty}\right)^N$; and $[0,1]$ is rescaled to be $[0,M]$. The compact mapping is defined as

$$T(M,\varphi,v) = (\rho,u) - (0,u_0),$$

where (ρ, u) satisfy

$$\left.\begin{array}{l} \alpha\rho^p + \text{div }(\rho v) - \varepsilon\Delta\rho = \frac{\alpha M}{|\Omega|}, \\ \frac{\alpha M}{|\Omega|}\frac{u}{2} + \rho v\cdot\nabla v + \frac{1}{2}\varepsilon\Delta\rho v - \mu\Delta u - \xi\nabla\text{div } u + a\nabla\rho^\gamma + \delta\nabla\rho^2 = \rho f + g, \end{array}\right\} \text{ in } \Omega,$$
$$\frac{\partial\rho}{\partial n} = 0, \quad u = 0, \qquad \text{on } \partial\Omega.$$

Notice that the compactness follows from the fact that $\cap_{1\leq q<\infty} W^{2,q} \hookrightarrow\hookrightarrow W^{1,\infty}$, and the uniform bounds in Condition 2 of Theorem 2 follows readily from the classical elliptic estimates in $W^{2,q}$, $1 \leq q < \infty$ and a bootstrap argument.

Step III: Passage to limits.

Before passing to limit $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, we recall

Lemma 3. *([1, Appendix D]). Let (E, d) be a complete metric space and $\{C_n\}$ be a sequence of continua (closed, connected subsets) in $E \times [0, \infty)$ with*

(A1) C_n is unbounded in $E \times \boldsymbol{R}$;

(A2) $\exists\ x_0 \in E,\ s.t.\ (x_0, 0) \in C_n$;

(A3) $C_n \cap (E \times [0, R]) \subset K_R$, K_R compact in $E \times \boldsymbol{R}$, $\forall\ R > 0$; or equivalently

(A3′) $C_n \cap (E \cap [0, R])$ is compact:

$$(x_n, t_n) \in C_n,\ t_n \text{ bounded} \Rightarrow x_n \text{ relatively compact in } E.$$

Then the limit continuum

$$C = \{(x, t) \in E \times [0, \infty);\ \exists\ \{n_k\},\ \exists\ x_{n_k} \to x,\ \exists\ t_{n_k} \to t,\ (x_{n_k}, t_{n_k}) \in C_{n_k}\}$$

satisfies

(C1) C is unbounded in $E \times \boldsymbol{R}$:

$$\forall\ t \geq 0,\ \exists\ x \in E,\ s.t.\ (x, t) \in C;$$

(C2) $(x_0, 0) \in C$;

(C3) $C \cap (E \times [0, R]) \subset K_{R'}$, $\forall\ R' > R \geq 0$.

We now commence our passage to limits, $\varepsilon \to 0_+$, then $\delta \to 0_+$, then $\alpha \to 0_+$, by invoking Lemma 3 to construct

$$C_\alpha^{\delta,\varepsilon} \to_\varepsilon C_\alpha^\delta \to_\delta C_\alpha \to_\alpha C\ \big(\text{this } C \text{ being what we pursue}\big).$$

1. $\varepsilon \to 0_+$**, for** $\alpha, \delta \in (0, 1]$ **fixed.**

The underlying $E = L^{q_1} \times \left(W^{1,q_2}\right)^N, \ 1 \le q_1 < 3, \ 1 \le q_2 < 2.$

(A1) holds since $\int_\Omega \rho^p = M.$

(A2) holds since $(0, u_0) \in C_\alpha^{\delta,\varepsilon}$.

(A3′) Let $0 < \varepsilon_n \to 0$, $0 \le M_n \to M$, $(\rho_n, u_n) \in C_\alpha^{\delta,\varepsilon_n}$. We show the compactness of (ρ_n, u_n) in E as

$\rho_n \rightharpoonup \rho \ge 0$ in L^3; $u_n \rightharpoonup u$ in H^1, $u_n \to u$ in $L^p (1 \le p < 6)$, $u_n \to u$ a.e.;

$$\nabla \left\{\text{div } u_n - \tfrac{a}{\mu+\xi}\rho_n^{5/3} - \tfrac{\delta}{\mu+\xi}\rho_n^2\right\} + \tfrac{\mu}{\mu+\xi}\text{curl curl } u$$
$$= (\rho u \cdot \nabla) u + \cdots \text{ bounded in } \left(L^3 \cdot L^6\right) \cdot L^2 \subset \mathcal{H}^1$$
$$\Rightarrow \nabla \left\{\text{div } u_n - \frac{a}{\mu+\xi}\rho_n^{5/3} - \frac{\delta}{\mu+\xi}\rho_n^2\right\}, \nabla\text{curl } u_n \text{ bounded in } \mathcal{H}^1$$
$$\Rightarrow \text{div } u_n - \frac{a}{\mu+\xi}\rho_n^{5/3} - \frac{\delta}{\mu+\xi}\rho_n^2 \text{ compact in } L^s\left(1 \le s < \frac{3}{2}\right); \text{curl } u_n \text{ compact in } L^r (1 \le r < 2)$$
$$\Rightarrow \rho_n \to \rho \text{ in } L^{q_1} (1 \le q_1 < 3)$$
$$\Rightarrow \text{div } u_n, \text{curl } u_n, \text{ and thus } Du_n \ \to \ \text{div } u, \text{curl } u, Du \text{ in } L^{q_2} (1 \le q_2 < 2), \text{ respectively.}$$

Thus we have a continuum C_α^δ of solutions of

$$\left.\begin{aligned} &\alpha\rho^p + \text{div } (\rho u) = \tfrac{\alpha M}{|\Omega|}, \\ &\alpha\rho^p u + \text{div } (\rho u \otimes u) - \mu\Delta u - \xi\nabla\text{div } u + a\nabla\rho^{5/3} + \delta\nabla\rho^2 = \rho f + g \end{aligned}\right\} \text{in } \Omega$$

satisfying (C1), (C2), (C3) in Lemma 3 and

$$C_\alpha^\delta \cap \{(\rho, u, M) ; \ 0 \le M \le R\} \text{ is bounded in } L^3 \times H_0^1 \times \boldsymbol{R}, \ \forall\, R > 0,$$

and the energy inequality

$$\int_\Omega \left\{\frac{\alpha}{2}h\,|u|^2 + \frac{1}{2}\alpha\rho^p\,|u|^2 + \mu\,|Du|^2 + \xi\,|\text{div } u|^2 + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} - h\rho^{\gamma-1}\right) + 2\delta\alpha\left(\rho^{p+1} - h\rho\right)\right\}$$
$$\le \int_\Omega \{\rho u \cdot f + u \cdot g\}, \ \forall \ (\rho, u, M) \in C_\alpha^\delta \left(h = \frac{\alpha M}{|\Omega|}\right).$$

2. $\delta \to 0_+$**, for** $\alpha \in (0, 1]$ **fixed.**

The space we live now is $E = L^q \times \left(W^{1,q}\right)^N$, $1 \le q < 2$. And the crucial key point is the compact assertion (A3′), which is proved as

$$\begin{aligned}
&\nabla\left\{\operatorname{div} u_n - \tfrac{a}{\mu+\xi}\rho_n^{5/3}\right\} + \tfrac{\mu}{\mu+\xi}\operatorname{curl}\operatorname{curl} u_n \\
&\quad = (\rho_n u_n \cdot \nabla) u_n + \cdots \text{ bounded in } \left(L^2 \cdot L^6\right)\cdot L^2 \subset \mathcal{H}^{6/7} \ \left(\text{by Step I}\right) \\
\Rightarrow\ & \begin{cases} \operatorname{div} u_n - \tfrac{a}{\mu+\xi}\rho_n^{5/3} \text{ compact in } L^s\left(1 \le s < \tfrac{6}{5}\right) \\ \operatorname{curl} u_n \text{ compact in } L^r (1 \le r < 2) \end{cases} \left(-1 + \frac{3}{6/7} = \frac{3}{6/5}\right) \\
\Rightarrow\ & \rho_n \to \rho \text{ in } L^q (1 \le q < 2) \\
\Rightarrow\ & \operatorname{div} u_n, \operatorname{curl} u_n, \text{ and thus } Du_n \text{ compact in } L^q (1 \le q < 2).
\end{aligned}$$

Thus we find a continuum of solutions of (3) satisfying (C1), (C2), (C3) and

$$C_\alpha \cap \{(\rho, u, M)\,;\ 0 \le M \le R\}$$

$$\text{is bounded in } L^{\max\{2, p+2/3\}} \times H_0^1 \times \boldsymbol{R},\ \forall\, R > 0,$$

and the energy inequality

$$\begin{aligned}
&\int_\Omega \left\{\frac{\alpha}{2} h\,|u|^2 + \frac{\alpha}{2}\rho^p\,|u|^2 + \mu\,|Du|^2 + \xi\,|\operatorname{div} u|^2 + \frac{a\alpha\gamma}{\gamma-1}\left(\rho^{\gamma+p-1} + h\rho^{\gamma-1}\right)\right\} \\
&\quad \le \int_\Omega \{\rho u \cdot f + u \cdot g\},\ \forall\ (\rho, u, M) \in C_\alpha \ \left(h = \frac{\alpha M}{|\Omega|}\right).
\end{aligned}$$

3. $\alpha \to 0_+$ **finally.**

The space we work in now is $E = L^p \times \left(W^{1,p}\right)^N$, $1 \le p < 2$. The details being exactly the same as the passage to limit $\delta \to 0_+$. And we conclude the existence of such a continuum C of solutions of (1) stated in Theorem 1.

□

REFERENCES

[1] P.L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models*. The Clarendon Press, Oxford University Press, New York, 1998.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA
E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume 1, Number 13, December 2010
Website: http://www.sciencenet.cn/u/zjzhang/
pp. 49-53

AN EXISTENCE THEOREM FOR STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS UNDER MODIFIED DIRICHLET BOUNDARY CONDITIONS

ZUJIN ZHANG

ABSTRACT. Four types of boundary conditions are considered for the stationary compressible Navier-Stokes equations. This is [1, Page 121], and is delivered on Dec. 11th, 2010.

1. **Introduction.** In this short paper, we consider the following stationary incompressible Navier-Stokes equations:

$$\left.\begin{array}{l}\mathrm{div}\ (\rho u) = 0\\ \mathrm{div}\ (\rho u \otimes u) - \mu\Delta u - \xi\nabla \mathrm{div}\ u + \nabla\,(a\rho^\gamma) = \rho f + g\end{array}\right\}\text{in } \Omega, \tag{1}$$

under boundary condition

(BC1) $u \cdot n = 0$ on $\partial\Omega$; or

(BC2) $\left.\begin{array}{l}\mathrm{curl}\ u = 0\ (N=2)\\ \mathrm{curl}\ u \times n = 0\ (N=3)\end{array}\right\}$ on $\partial\Omega$; or

(BC3) $(d \cdot n + Au) \times n = 0$ on $\partial\Omega$, with

$$d = \frac{\nabla u + (\nabla u)^t}{2} \text{ is the deformation tensor,}$$

$$A \text{ is a positive-definite matrix,}$$

$$(Qx + u_0) \cdot n(x) \neq 0 \text{ on } \partial\Omega,\ \forall \text{ antisymmetric } N \times N \text{ matrix } Q \text{ and } u_0 \in \boldsymbol{R}^N,\ \text{unless } Q = 0, u_0 = 0,$$

$$\xi > \frac{N-2}{N}\mu;\ \text{or}$$

Key words and phrases. existence of a solution, compressible Navier-Stokes equations, modified Dirichlet boundary value problem.

(BC4) $\left(\frac{\partial u}{\partial n} + Au\right) \times n = 0$ on $\partial\Omega$, with

A is a nonpositive-definite (not necessarily symmetric) matrix,

$$\xi \geq -\frac{\mu}{N}.$$

2. **Existence Result.** The main result now reads

Theorem 1. *Let $N = 2$ or $N = 3$, $\gamma > 0$, and $p = p(\gamma, N)$ is large enough. Then there exists a continuum $C\left(\subset L^q \times W^{1,q},\ 1 \leq q < \infty\right)$ of solutions of (1) under (BC1), or (BC2), or (BC3), or (BC4), satisfying*

1. *$(0, u_0) \in C$, with u_0 solves*

$$\begin{cases} -\mu\Delta u_0 - \xi\nabla div\ u_0 = 0, & in\ \Omega, \\ u_0 \text{ satisfies (BC1), or (BC2), or (BC3), or (BC4).} \end{cases}$$

2. *$\forall\ M \in [0, \infty)$, $\exists\ (\rho, u) \in C$, such that $\int \rho^p = M$.*

Proof. 1. We approximate (1) by

$$\begin{cases} \text{div } (\rho u) = 0,\ \rho \geq 0,\ \text{in } \Omega,\ \int_\Omega \rho^p = M, \\ \text{div } (\rho u \otimes u) - \mu\Delta u - \xi\nabla\text{div } u + \nabla\left(a\rho^\gamma + \alpha\rho^p\right) = \rho f + g,\ \text{in } \Omega, \\ u \text{ satisfies (BC1), or (BC2), or (BC3), or (BC4),} \end{cases} \tag{2}$$

with $\alpha \in (0, 1]$, and $p > 3$ is large enough.

2. Notice that the proof of
   (a) the existence of a solution continuum C_α to (2); and
   (b) the passage to limit $C_\alpha \to_\alpha C$;
   are exactly the same as in [2].
3. Thus we need only to show a prior that

$$\left.\begin{array}{l} (\rho, u, M) \in C_\alpha \\ 0 \leq M \leq R < \infty \end{array}\right\} \Rightarrow \begin{cases} \rho \text{ bdd in } L^\infty,\ u \text{ bdd in } W^{1,q}, \\ \text{div } u - \frac{a}{\mu+\xi}\rho^\gamma - \frac{\alpha}{\mu+\xi}\rho^p \text{ bdd in } W^{1,q},\ \ \textbf{uniformly in } \alpha \in (\mathbf{0}, \mathbf{1}]. \\ \text{curl } u \text{ bdd in } W^{1,q},\ \forall\ 1 \leq q < \infty, \end{cases}$$

For this purpose, we shall consider $N = 3$ ($N = 2$ being similar and simple). Our strategy is the usual (by now) **bootstrap argument involving the Hodge decomposition**.

Write (2)$_2$ in the form

$$\nabla\left\{\operatorname{div} u - \frac{a}{\mu+\xi}\rho^\gamma - \frac{\alpha}{\mu+\xi}\rho^p\right\} + \frac{\mu}{\mu+\xi}\operatorname{curl}\operatorname{curl} u = (\rho u\cdot\nabla)\,u + \cdots . \tag{3}$$

We use (3) to bootstrap the regularity of u, and then that of ρ by (2)$_2$. Take first $\rho \in L^{p_i}$, $\nabla u \in L^{q_i}$, with $p_0 = p$, $q_0 = 2$, we have

$$\nabla\left\{\operatorname{div} u - \frac{a}{\mu+\xi}\rho^\gamma - \frac{\alpha}{\mu+\xi}\rho^p\right\},\ D\operatorname{curl} u\ \in L^{r_i},\ \frac{1}{r_i} = \frac{1}{p_i} + \left(\frac{1}{q_i} - \frac{1}{3}\right) + \frac{1}{q_i};$$

$$Du,\ a\rho^\gamma + \alpha\rho^p\ \in L^{q_{i+1}},\ \frac{1}{q_{i+1}} = \frac{1}{r_i} - \frac{1}{3} = \frac{1}{p_i} + \frac{2}{q_i} - \frac{2}{3}\ \big(\text{by } (4)\big).$$

Notice that $p_{i+1} = p_i = p$, since we want to get the uniform bounds (independent of α). Thus

$$\begin{aligned}\frac{1}{q_{i+1}} &= \frac{1}{p_i} + \frac{2}{q_i} - \frac{2}{3} = 2^{i+1}\frac{1}{q_0} + \left(\frac{1}{p} - \frac{2}{3}\right)\left(1 + 2 + \cdots + 2^i\right)\\ &= 2^i + \left(\frac{1}{p} - \frac{2}{3}\right)(2^{i+1} - 1) = 2^i\left[-2\left(\frac{2}{3} - \frac{1}{p}\right) + 1\right] + \frac{2}{3} - \frac{1}{p}\\ &< \frac{1}{3},\ \text{if } i \text{ large.}\end{aligned}$$

Hence $Du \in L^{q_{i+1}>3} \Rightarrow u \in L^\infty$. From then on, we may bootstrap as

$$\frac{1}{q_{i+1}} = \left(\frac{1}{p} + \frac{1}{q_i}\right) - \frac{1}{3} = \frac{1}{q_0} - i\left(\frac{1}{p} - \frac{1}{3}\right) = \frac{1}{2} - i\left(\frac{1}{p} - \frac{1}{3}\right) < 0,\ \text{if } i \text{ large.}$$

Consequently, $Du \in L^q,\ 1 \le q < \infty$, and

$$\nabla\left(a\rho^\gamma + \alpha\rho^p\right) = \cdots \text{ by } (2)_2 \Rightarrow \nabla\left(a\rho^\gamma + \alpha\rho^p\right) \in L^q,\ 1 \le q < \infty.$$

□

Remark 2. *One may use many variants for the approximation of the stationary problem* (1)*, other than* (2)*, or those in* [2].

Remark 3. *As we know, for* (1)*,*

1. *when* $M = 0$, *there exists an unique solution* u *of* (1);
2. *however, for* $M > 0$ *small, we do not have uniqueness of solutions of* (1), *see* [1, Remark 6.16, Page 117].

Thus, the existence result for small $M > 0$ *could not be obtained by invoking (variants of) implicit function theorem (to yield an unique branch of solutions).*

3. A technical Lemma.

Lemma 4. *Let*

1. $0 \leq \rho \in L^p(\Omega)$, $1 \leq p \leq \infty$;
2. $u \in W^{1,q}(\Omega)$, $1 \leq q \leq \infty$ *with* $u \cdot n = 0$ *on* $\partial\Omega$;
3. $\frac{1}{p} + \frac{1}{q} \leq 1$; *and*
4. *div* $(\rho u) = 0$ *in* Ω.

Then

$$\|\varphi(\rho)\|_r \leq \left\| div\ u - \varphi(\rho) \right\|_r, \ \forall\ \varphi \in C([0,\infty)). \tag{4}$$

Proof. We just prove (4) formally, with the verification being direct consequence of regularizations.

$$\begin{aligned}
& \operatorname{div}\ (\rho u) = 0 \\
\Rightarrow\ & \operatorname{div}\ [\beta(\rho) u] = u \cdot \nabla\beta(\rho) + \beta(\rho)\operatorname{div} u = u\nabla\beta(\rho) + \frac{\beta(\rho)}{\rho}[-u\cdot\nabla\rho] = \left[\beta'(\rho) - \frac{\beta(\rho)}{\rho}\right] u \cdot \nabla\rho \\
\Rightarrow\ & u \cdot \nabla\varphi(\rho) = \operatorname{div}\ [\beta(\rho)u] \text{ for } \varphi'(\rho) = \beta'(\rho) - \frac{\beta(\rho)}{\rho} \\
& \left(t\beta'(t) - \beta(t) = t\varphi'(t) \Rightarrow \left[\tilde\beta(s) = \beta(e^s)\right]\tilde\beta'(s) - \tilde\beta(s) = e^s\varphi'(e^s)\right) \\
\Rightarrow\ & 0 = \int_\Omega \operatorname{div}\ [\beta(\rho)u] = \int_\Omega u \cdot \varphi(\rho) = -\int_\Omega \varphi(\rho)\operatorname{div} u \\
\Rightarrow\ & \int_\Omega |\varphi(\rho)|^p = \int_\Omega [\varphi(\rho) - \operatorname{div} u]\,|\varphi(\rho)|^{p-2}\,\varphi(\rho) \leq \|\varphi(\rho) - \operatorname{div} u\|_p\,\|\varphi(\rho)\|_p^{p-1} \\
\Rightarrow\ & \|\varphi(\rho)\|_p \leq \|\varphi(\rho) - \operatorname{div} u\|_p\,.
\end{aligned}$$

□

REFERENCES

[1] P.L. Lions, *M*athematical topics in fluid mechanics. Vol. 2. Compressible models. The Clarendon Press, Oxford University Press, New York, 1998.

[2] Z.J. Zhang, *A*n existence theorem for stationary compressible Navier-Stokes equations with Dirichlet boundary conditions. Charleton Univ. Math. J., **12** (2010) 43-48.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume 1, Number 14, December 2010

Website: http://www.sciencenet.cn/u/zjzhang/
pp. 54-68

EXTERIOR PROBLEMS AND RELATED QUESTIONS FOR THE STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS

ZUJIN ZHANG

ABSTRACT. In this paper, we consider the stationary compressible Navier-Stokes equations either in the whole space, in the exterior domain, or in a tube. Various existence resulsts are obtained. This is [1, Sect. 6.8], and is delivered on Dec. 11th & 18th & 25th, 2010.

CONTENTS

Key words and phrases. existence of a solution, compressible Navier-Stokes equations, exterior domain, tube domain, invading domain technique, vanishing damping technique.

1. **Introduction.** In this paper, we consider the following stationary compressible Navier-Stokes equations:

$$\begin{cases} \text{div } (\rho u) = 0, \quad \rho \geq 0, \\ \text{div } (\rho u \otimes u) - \mu \Delta u - \xi \nabla \text{div } u + a \nabla \rho^{\gamma} = \rho f + g, \end{cases} \tag{1}$$

in

1. either (whole space case) $\Omega = \boldsymbol{R}^N$;
2. (exterior case) $\Omega = \omega^c$, with ω a bounded smooth connected domain in $\boldsymbol{R}^N$; or
3. (tube case) $\Omega = \boldsymbol{R} \times \omega$, with ω a bounded smooth connected domain in $\boldsymbol{R}^{N-1}$.

We couple (1) with physical relevant boundary conditions (that is, the flow is constant at infinity),

1. in the whole space case,

$$\rho \to \rho^{\infty}, \quad u \to u^{\infty}, \quad \text{as } |x| \to \infty;$$

2. in the exterior case,

$$\begin{cases} \rho \to \rho^{\infty}, \quad u \to u^{\infty}, \quad \text{as } |x| \to \infty, \\ u|_{\partial\Omega} = 0; \end{cases}$$

3. in the tube case,

$$\rho \to \rho^{\infty}_{\pm}, \quad u \to 0, \quad \text{as } x_1 \to \pm\infty.$$

Notice that

1. If we insist the behavior of ρ, u at infinity to be zero, we would rather obtain the trivial solution, see [1, Remark 6.2].
2. If we insist $u^{\infty} \neq 0$, then the problem is closed related to the "inflow" open problem, see Sect. 4. We shall investigate this issue in a forthcoming paper.
3. If we insist $\rho^{\infty}_{+} \neq \rho^{\infty}_{-}$ in the tube case, we may construct however a non-existence result. In fact, assuming

$$\rho^{\infty}(\Omega), \; Du \in L^2(\Omega), \; u \in L^2 \cap L^{\infty}(\Omega),$$
$$g = 0, \; f = 0 \text{ or } \nabla \Phi \text{ with } \Phi \text{ smooth, vanishing fast enough as } |x_1| \to \infty,$$

and taking the inner product of (1)$_2$ with u in $L^2((-R,R)\times\omega)$, $0<R<\infty$, we find

$$\int_\omega dx' \left\{\left[\frac{1}{2}\rho|u|^2+\frac{a\gamma}{\gamma-1}\rho^\gamma\right]_{x_1=-R}^{x_1=R}-\left[\mu u\cdot\frac{\partial u}{\partial x_1}+\xi u\text{div } u\right]_{x_1=-R}^{x_1=R}\right\}$$
$$+\int_{-R}^R\int_\omega dx'\left\{\mu|Du|^2+\xi|\text{div } u|^2\right\}=0.$$

Sending $R\to\infty$, we deduce

$$\rho_+^\infty\le\rho_-^\infty.$$

Z In view of the aforementioned considerations, we shall concentrate ourselves investigating (1) under boundary conditions stated above with

$$\begin{cases}\rho^\infty>0\ or\ (\rho_+^\infty=\rho_-^\infty=\rho^\infty>0),\\ u^\infty=0.\end{cases}$$

We end this introduction by outlining the rest of this paper. In Sect. 2, we consider (1) in $\Omega=\boldsymbol{R}^N$. Three existence results are established. Section 3 is devoted to extending the existence results in Sect. 2 to the exterior or tube cases. And finally, an "inflow" open problem is polished in Sect. 4.

2. **Stationary compressible Navier-Stokes equations in the whole space.** Detailed in this section are various existence results for the problem

$$\begin{cases}\text{div }(\rho u)=0,\ \rho\ge0, & \text{in } \boldsymbol{R}^N,\\ \text{div }(\rho u\otimes u)-\mu\Delta u-\xi\nabla\text{div }\ u+a\nabla\rho^\gamma=\rho f+g, & \text{in } \boldsymbol{R}^N,\\ \rho\to\rho^\infty>0,\ u\to0, & \text{as } |x|\to\infty.\end{cases}\qquad(2)$$

Here we assume

1. $g\in L^1\cap L^\infty(\boldsymbol{R}^N)$ for simplicity;
2. $N\ge3$ for convenience (to ensure the decay of Green's function for second-order elliptic operators, see Remark 3).

2.1. (2) with general force f.

Theorem 1. *Let $N \geq 3$, $\gamma > \max\left\{3, \frac{N}{2}\right\}$. Then there exists a solution (ρ, u) of (2) such that*

$$\rho - \rho^\infty \in \begin{cases} L^3 \cap L^\infty(\boldsymbol{R}^N), & \text{if } N = 3, \\ L^2 \cap L^{\frac{N}{N-2}(\gamma-1)}(\boldsymbol{R}^N), & \text{if } N \geq 4; \end{cases} \qquad \nabla u \in L^2(\boldsymbol{R}^N), \quad u \in L^{\frac{2N}{N-2}}(\boldsymbol{R}^N).$$

Remark 2. *If $N = 3$, $\gamma > 3$, we have $\rho \in L^\infty(\boldsymbol{R}^N)$, and may bootstrap using the Hodge decomposition the regularity of*

$$div\ u - \frac{a}{\mu + \xi}\rho^\gamma, \quad curl\ u$$

to be in

$$W^{1,q}_{unif}(\boldsymbol{R}^N) = \left\{\varphi \in W^{1,q}_{loc}(\boldsymbol{R}^N);\ \sup_{y \in \boldsymbol{R}^N} \int_{y+B_1} |\varphi|^q + |D\varphi|^q < \infty\right\}, \quad \forall\ 1 \leq q < \infty,$$

and $Du \in BMO(\boldsymbol{R}^N)$.

Remark 3. *The behavior at infinity of (ρ, u) is not clear. However, the best possible decay at infinity is:*

$$|\rho(x) - \rho^\infty| \leq \frac{C}{|x|^{\frac{N-1}{\gamma-1}}}, \qquad |u(x)| \leq \frac{C}{|x|^{N-2}}, \qquad |Du(x)| \leq \frac{C}{|x|^{N-1}}.$$

In fact,

1. *if we take $f = \nabla\Phi \in L^1 \cap L^\infty(\boldsymbol{R}^N)$, $\Phi \in L^{\frac{N}{N-1}}(\boldsymbol{R}^N)$, $g \equiv 0$, $u \equiv 0$, then*

$$a\nabla\rho^\gamma = \rho\nabla\Phi \Rightarrow \rho^{\gamma-1} - (\rho^\infty)^{\gamma-1} = \frac{\gamma-1}{a\gamma}\Phi;$$

2. *if $\rho = \rho^\infty$, $f \equiv 0$, $g \in C_0^\infty(\boldsymbol{R}^N)$, $div\ g = 0$, then u solves*

$$-\mu\Delta u - \xi\nabla div\ u = g,$$

thus u decays at most like $\frac{1}{|x|^{N-2}}$, Du decays at most like $\frac{1}{|x|^{N-1}}$.

Hence, it is natural to conjecture that $\left(\frac{1}{|x|^q} \in L^{\frac{N}{q},\infty}\right)$

$$\rho - \rho^\infty \in L^{\frac{N(\gamma-1)}{N-1},\infty}(\boldsymbol{R}^N), \quad u \in L^{\frac{N}{N-2},\infty}(\boldsymbol{R}^N), \quad Du \in L^{\frac{N}{N-1},\infty}(\boldsymbol{R}^N).$$

Proof of Theorem 1. Our proof involves the **invading domain** and **vanishing damping** techniques, and is divided into five steps.

Step I: Formal a priori estimates.

First, multiplying $(2)_2$ by u, we obtain the usual local energy identity (by $(2)_1$):

$$\begin{aligned}&\mathrm{div}\ \left\{u\left[\rho\frac{|u|^2}{2}+\frac{a\gamma}{\gamma-1}\left(\rho^\gamma-(\rho^\infty)^{\gamma-1}\rho\right)\right]\right\}\\&-\mu\Delta\frac{|u|^2}{2}+\mu\,|Du|^2-\xi\mathrm{div}\ (u\mathrm{div}\ u)+\xi\,|\mathrm{div}\ u|^2=\rho u\cdot f+u\cdot g,\ \text{in}\ \boldsymbol{R}^n.\end{aligned}$$

Integrating then over $\boldsymbol{R}^n$ (taking into account of $(2)_3$), we deduce

$$\int_{\boldsymbol{R}^N}\left[\mu\,|Du|^2+\xi\,|\mathrm{div}\ u|^2\right]=\int_{\boldsymbol{R}^N}[\rho u\cdot f+u\cdot g]\,. \tag{3}$$

Using Sobolev inequality and assumptions of f, g, we have

$$\|u\|_{\frac{2N}{N-2}}+\|Du\|_2\leq C\left(1+\|\rho\|_{\infty+q}\right),$$

where q is specified later on.

Second, taking the divergence of $(2)_2$, and using $(2)_3$, we see

$$a\rho^\gamma=a\,(\rho^\infty)^\gamma+(\mu+\xi)\,\mathrm{div}\ u+R_iR_j\left(\rho u_iu_j\right)-(-\Delta)^{-1}\,\mathrm{div}\ (\rho f+g)\,,\ \text{in}\ \boldsymbol{R}^N; \tag{4}$$

$$\|\rho^\gamma\|_{\infty+\frac{q}{\gamma}}\leq C\left[1+\|Du\|_2+\|\rho\|_{\infty+q}\,\|u\|^2_{\frac{2N}{N+2}}+\|\rho\|_{\infty+q}\right]; \tag{5}$$

with

1. $\frac{q}{\gamma}\leq 2<\infty$ ($L^q\subset L^{q_1}+L^{q_2},\ q_1\leq q\leq q_2$);
2. $\frac{1}{q}+\frac{N-2}{N}\leq\frac{\gamma}{q}$:

$$\begin{aligned}\left\|\rho\,|u|^2\right\|_{\infty+\frac{q}{\gamma}}&\leq\left\|\rho\,|u|^2\right\|_{\frac{N}{N+2},1/\left(\frac{1}{q}+\frac{N-2}{N}\right)}\ (L^{p_1}+L^{p_2}\subset L^{q_1}+L^{q_2},\ q_1\leq p_1\leq p_2\leq q_2)\\&\leq\|\rho\|_{\infty+q}\,\|u\|^2_{\frac{2N}{N-2}}\,.\end{aligned}$$

Recalling the bounds of solutions to the discretized stationary compressible Navier-Stokes equations (see [1, Theorem 6.1]), we set

$$q=\begin{cases}2\gamma, & \text{if}\ N=3,\\ \frac{N}{N-2}(\gamma-1), & \text{if}\ N\geq 4.\end{cases} \tag{6}$$

Combining (3) with (5), we gather

$$\|\rho\|_{\infty+q}^{\gamma} \leq C\left(1 + \|\rho\|_{\infty+q}^{\frac{2\gamma}{\gamma-1}}\right).$$

Since $\gamma > 3$, we have consequently the following a priori bounds on

1. ρ in $L^\infty + L^q$, u in $L^{\frac{2N}{N-2}}$, Du in L^2;
2. when $N = 3$, by (4), $\rho^\gamma - (\rho^\infty)^\gamma$ in $L^2 + L^3$:

$$\rho\,|u|^2 \in (L^\infty + L^q)\cdot L^3 = L^3 + L^{1/\left(\frac{1}{2\gamma}+\frac{1}{3}\right)} \subset L^3 + L^2;$$

   using the bootstrap argument involving Hodge decomposition (see [1, Theorem 6.3 and its proof on Page 71]), ρ in L^∞;
   thus $\rho - \rho^\infty$ in $\left(L^2 + L^3\right) \cap L^\infty = L^3 + L^\infty$:

$$\begin{cases} |\rho^\gamma - (\rho^\infty)^\gamma|\cdot 1_{|\rho-\rho^\infty|\leq\frac{\rho^\infty}{2}} = \gamma\xi^{\gamma-1}\,|\rho-\rho^\infty|\cdot 1_{|\rho-\rho^\infty|\leq\frac{\rho^\infty}{2}} & \\ \qquad\qquad \geq C\,|\rho-\rho^\infty|\cdot 1_{|\rho-\rho^\infty|\leq\frac{\rho^\infty}{2}} & \Rightarrow\ (\rho-\rho^\infty)\cdot 1_{|\rho-\rho^\infty|\leq\frac{\rho^\infty}{2}} \in L^2 + L^3, \\ \rho-\rho^\infty \in L^{2\gamma} + L^{3\gamma}\ \&\ \text{Lemma 5} & \Rightarrow\ (\rho-\rho^\infty)\cdot 1_{|\rho-\rho^\infty|>\frac{\rho^\infty}{2}} \in L^1, \end{cases}$$

   and by invoking Lemma 4 just before Step 2;
3. when $N \geq 4$, by (4), $\rho^\gamma - (\rho^\infty)^\gamma$ in $L^2 + L^{\frac{N}{N-2}\frac{\gamma-1}{\gamma}}$:

$$\rho\,|u|^2 \in (L^\infty + L^q)\cdot L^{\frac{N}{N-2}} = L^{\frac{N}{N-2}} + L^{\frac{N}{N-2}\frac{\gamma-1}{\gamma}},$$

   thus $\rho - \rho^\infty$ in $\left(L^1 + L^2\right) \cap \left(L^{\frac{N}{N-2}(\gamma-1)} + L^\infty\right) = L^2 \cap L^{\frac{N}{N-2}(\gamma-1)}$ by the same reasoning as the case when $N = 3$, and by using Lemma 4 just below.

We now state and prove some **technical** lemmas we have utilized.

Lemma 4. *1. Let $1 \leq a \leq b \leq \infty$, $1 \leq c \leq \infty$. Then*

$$\left(L^a + L^b\right) \cap L^c = \begin{cases} L^c \cap L^a, & \text{if } c \leq a, \\ L^c, & \text{if } a \leq c \leq b, \\ L^b \cap L^c, & \text{if } c \geq b. \end{cases} \tag{7}$$

2. Let $1 \leq a \leq b \leq \infty$, $\leq c \leq d \leq \infty$. Then

$$\left(L^a + L^b\right) \cap \left(L^c + L^d\right) = \begin{cases} L^b \cap L^c, & \text{if } b \leq c, \\ L^c + L^d, & \text{if } a \leq c \leq d \leq b. \end{cases} \tag{8}$$

Proof. 1. **Claim.** If $f \in L^a + L^b$, then

$$\exists\, 0 \le g_1 \in L^a,\ 0 \le g_2 \in L^b,\ s.t.\ |f| = g_1 + g_2.$$

In fact, if $f = f_1 + f_2 \in L^a + L^b$, then

$$|f| = \min\{|f|, |f_1|\} + (|f| - |f_1|)_+,$$

with $0 \le \min\{|f|, |f_1|\} \le |f_1| \in L^a$, $0 \le (|f| - |f_1|)_+ \le |f_2| \in L^b$.

2. We now prove (7). If $f \in \left(L^a + L^b\right) \cap L^c$, then

$$0 \le g_1, g_2 \le |f| \Rightarrow g_1 \in L^a \cap L^c,\ g_2 \in L^b \cap L^c.$$

(a) If $c \le a$, then $g_2 \in L^a \cap L^c$, $|f| = g_1 + g_2 \in L^a \cap L^c$.

(b) If $c \ge b$, then $g_1 \in L^b \cap L^c$, $|f| = g_1 + g_2 \in L^b \cap L^c$.

(c) If $a \le c \le b$, then $f \in L^c \Rightarrow |f| = |f| \cdot 1_{|f|<1} + |f| \cdot 1_{|f|\ge 1} \in L^a + L^b$.

3. We next show (8). If $f \in \left(L^a + L^b\right) \cap \left(L^c + L^d\right)$, then

$$0 \le g_1, g_2 \le |f| \Rightarrow g_1, g_2 \in L^c + L^d \Rightarrow g_1 \in L^a \cap \left(L^c + L^d\right),\ g_2 \in L^b \cap \left(L^c + L^d\right).$$

(a) If $b \le c$, then $g_1 \in L^a \cap L^c \subset L^b \cap L^c$, $g_2 \in L^b \cap L^c$, $|f| = g_1 + g_2 \in L^b + L^c$.

(b) If $a \le c \le d \le b$, then $L^c + L^d \subset L^a + L^b$.

□

Lemma 5. *Let $f \in L^p + L^q$, $1 \le p < q < \infty$. Then*

$$f \cdot 1_{|f|\ge t} \in L^1,\ \forall\, 0 < t < \infty. \tag{9}$$

Proof. As the claim in the proof of Lemma 4 shows

$$\exists\, 0 \le g_1 \in L^p,\ 0 \le g_2 \in L^q,\ s.t.\ |f| = g_1 + g_2.$$

Hence

$$\{|f| \ge t\} \subset \left\{g_1 \ge \frac{t}{2}\right\} \cup \left\{g_2 \ge \frac{t}{2}\right\},$$

and consequently,

$$\int |f| \cdot 1_{|f|\ge t} \le \int (g_1 + g_2) \cdot \left(1_{g_1 \ge \frac{t}{2}} + 1_{g_2 \ge \frac{t}{2}}\right)$$

$$\leq \int g_1 \cdot 1_{g_1 \geq \frac{t}{2}} + \int g_1 \cdot 1_{g_2 \geq \frac{t}{2}} + \int g_2 \cdot 1_{g_1 \geq \frac{t}{2}} + \int g_2 \cdot 1_{g_2 \geq \frac{t}{2}}$$
$$\equiv I_1 + I_2 + I_3 + I_4. \tag{10}$$

The terms I_1, I_4 are treated similarly as

$$I_1 = \int g_1 \cdot 1_{g_1 \geq \frac{t}{2}} \leq \left(\frac{2}{t}\right)^{p-1} \int g_1^p,$$

$$I_4 = \int g_2 \cdot 1_{g_2 \geq \frac{t}{2}} \leq \left(\frac{2}{t}\right)^{q-1} \int g_2^q.$$

Meanwhile, I_2, I_3 are dominated by Hölder inequality as

$$I_2 = \int g_1 \cdot 1_{g_2 \geq \frac{t}{2}} \leq \left(\int g_1^p\right)^{\frac{1}{p}} \cdot \left|\left\{g_2 \geq \frac{t}{2}\right\}\right|^{\frac{p-1}{p}} \leq \|g_1\|_p \left[\frac{2}{t}\|g_2\|_q\right]^{\frac{q(p-1)}{p}},$$

$$I_3 = \int g_2 \cdot 1_{g_1 \geq \frac{t}{2}} \leq \left(\int g_2^q\right)^{\frac{1}{q}} \cdot \left|\left\{g_1 \geq \frac{t}{2}\right\}\right|^{\frac{q-1}{q}} \leq \|g_2\|_q \left[\frac{2}{t}\|g_1\|_p\right]^{\frac{p(q-1)}{q}}.$$

Gathering the last four displayed inequalities, (10) becomes

$$\int |f| \cdot 1_{|f| \geq t} \leq \left(\frac{2}{t}\right)^{p-1} \int g_1^p + \left(\frac{2}{t}\right)^{q-1} \int g_2^q + \|g_1\|_p \left[\frac{2}{t}\|g_2\|_q\right]^{\frac{q(p-1)}{p}} + \|g_2\|_q \left[\frac{2}{t}\|g_1\|_p\right]^{\frac{p(q-1)}{q}} < \infty.$$

□

Remark 6. *If $f \in L^p + L^\infty$, $1 \leq p < \infty$, then checking the proof above yields*

$$f \cdot 1_{|f| \geq 2\|g_2\|_\infty} \in L^1, \tag{11}$$

for example. We will use this fact later in Step II.

Sept II: Approximate problems and uniform bounds independent of $R \in (0, \infty)$.

We now approximate (2) by

$$\begin{cases} \alpha\rho + \text{div } (\rho u) = \alpha\rho^\infty, \rho \geq 0, & \text{in } B_R, \\ \alpha\rho u + \text{div } (\rho u \otimes u) - \mu\Delta u - \xi\nabla\text{div } u + a\nabla\rho^\gamma = \rho f + g, & \text{in } B_R, \\ u = 0, & \text{on } \partial B_R, \end{cases} \tag{12}$$

where $\alpha \in (0, 1]$, $R \in (0, \infty)$. The existence of a solution pair $(\rho_{\alpha,R}, u_{\alpha,R})$ of (12) is well known (see [1, Sect. 6.2]).

We shall establish the uniform bounds of the solution independent of $R \in (0, \infty)$, so that we may pass to limit $R \to \infty$ in Step III.

For convenience of notations, we omit the subscript α, R in the solution pair $(\rho_{\alpha,R}, u_{\alpha,R})$ in this step.

1. Bounds depending on $R \in (0, \infty)$—Energy-type.

$$\begin{cases} \rho \in L^\infty_{loc}(B_R) \cap L^{2\gamma}(B_R),\ u \in W^{2,p}_{loc}(B_R),\ \forall\ 1 \le p < \infty, & \text{if } N = 3,\\ \rho \in L^{\frac{N}{N-2}(\gamma-1)}(B_R), & \text{if } N = 4. \end{cases}$$

2. Bounds independent of $R \in (0, \infty)$.

$$\int_{B_R} \rho = \rho^\infty |B_R|;$$

$$\int_{B_R} \left\{ \frac{\alpha}{2}\rho^\infty |u|^2 + \frac{\alpha}{2}\rho |u|^2 + \mu |Du|^2 + \xi |\mathrm{div}\, u|^2 + \frac{a\gamma}{\gamma-1}\left(\rho^\gamma - \rho^{\gamma-1}\rho^\infty\right) \right\} \le \int_{B_R} \{\rho u \cdot f + u \cdot g\};$$

$$\begin{aligned}
\int_{B_R} \left\{\rho^\gamma - \rho^{\gamma-1}\rho^\infty\right\} &= \int_{B_R} \left\{\rho^\gamma - \rho^{\gamma-1}\rho^\infty + (\rho^\infty)^\gamma - (\rho^\infty)^{\gamma-1}\rho\right\}\\
&= \int_{B_R} \left\{\rho^{\gamma-1} - (\rho^\infty)^{\gamma-1}\right\}(\rho - \rho^\infty)\ (\ge 0)\\
&= \int_{B_R} \left\{\rho^{\gamma-1} - (\rho^\infty)^{\gamma-1}\right\}(\rho - \rho^\infty) \cdot \left(1_{\rho \le 2\rho^\infty} + 1_{\rho > 2\rho^\infty}\right)\\
&\ge \nu \left[\int_{B_R} |\rho - \rho^\infty|^2 \cdot 1_{\rho\le 2\rho^\infty} + \int_{B_R} \rho^\gamma \cdot 1_{\rho > 2\rho^\infty}\right]
\end{aligned}$$

$$\left(\begin{array}{l} \rho > 2\rho^\infty \Rightarrow \rho - \rho^\infty > \frac{\rho}{2},\ \rho^{\gamma-1} - (\rho^\infty)^{\gamma-1} > \left(1 - \frac{1}{2^{\gamma-1}}\right)\rho^{\gamma-1}\\ \frac{\rho^\infty}{2} \le \rho \le 2\rho^\infty \Rightarrow \rho^{\gamma-1} - (\rho^\infty)^{\gamma-1} = (\gamma-1)\xi^{\gamma-2}(\rho - \rho^\infty), \xi \in \left(\frac{\rho^\infty}{2}, 2\rho^\infty\right)\\ 0 \le \rho < \frac{\rho^\infty}{2} \Rightarrow \rho^\infty \ge \rho^\infty - \rho \ge \frac{\rho^\infty}{2}, (\rho^\infty)^{\gamma-1} - \rho^{\gamma-1} \ge \left(1 - \frac{1}{2^{\gamma-1}}\right)(\rho^\infty)^{\gamma-1} \end{array}\right);$$

$$\begin{aligned}
\int_{B_R} \{\rho u \cdot f + u \cdot g\} &= \int_{B_R} \left\{\left[(\rho - \rho^\infty) \cdot 1_{\rho\le 2\rho^\infty} + \rho^\infty \cdot 1_{\rho \le 2\rho\infty} + \rho \cdot 1_{\rho > 2\rho^\infty}\right] u \cdot f + u \cdot g\right\}\\
&\le \left\|(\rho - \rho^\infty) \cdot 1_{\rho\le 2\rho^\infty}\right\|_\infty \|u\|_{\frac{2N}{N-2}} \|f\|_{\frac{2N}{N+2}} + \left\|\rho^\infty \cdot 1_{\rho\le 2\rho^\infty}\right\|_\infty \|u\|_{\frac{2N}{N-2}} \|f\|_{\frac{2N}{N+2}} + \left\|\rho \cdot 1_{\rho > 2\rho^\infty}\right\|_\gamma \|u\|_{\frac{2N}{N-2}} \|f\|_a\\
&\left(\frac{1}{\gamma} + \left(\frac{1}{2} - \frac{1}{N}\right) + \frac{1}{a} = 1\right)\\
&\varepsilon \|Du\|_2^2 + C\|f\|_N^2 + C\|f\|^2_{\frac{2N}{N+2}} + \varepsilon \left\|\rho \cdot 1_{\rho > 2\rho^\infty}\right\|_\gamma^\gamma + C\|f\|_a^{\frac{2\gamma}{\gamma-2}};
\end{aligned}$$

$$\int_{B_R}\left\{\frac{\alpha}{2}\rho^\infty|u|^2+\frac{\alpha}{2}\rho|u|^2+\mu|Du|^2+\xi|\mathrm{div}\ u|^2\right\}+\int_{B_R}\left\{|\rho-\rho^\infty|^2\cdot 1_{\rho\le 2\rho^\infty}+\rho^\gamma\cdot 1_{\rho>2\rho^\infty}\right\}\le C.$$

Thus

$$\begin{cases} u \text{ is bounded in } H_0^1(B_R), \\ \rho|u|^2,\ \left(\rho^{\gamma-1}-(\rho^\infty)^{\gamma-1}\right)(\rho-\rho^\infty) \text{ are bounded in } L^1, \\ (\rho-\rho^\infty)^2\cdot 1_{\rho\le 2\rho^\infty},\ \rho\cdot 1_{\rho>2\gamma^\infty} \text{ are bounded in } L^1, \\ \rho=\rho\cdot 1_{\rho\le 2\rho^\infty}+\rho\cdot 1_{\rho>2\rho^\infty} \text{ is bounded in } L^\infty+L^\gamma. \end{cases}$$

3. Bounds independent of $R\in(0,\infty)$—higher regularity.
   (a) $\fint_{B_R}\rho^\gamma$ is bounded in L^∞.

$$\fint_{B_R}\rho^\gamma\ge\left(\fint_{B_R}\rho\right)^\gamma=(\rho^\infty)^\gamma;$$

$$\fint_{B_R}\rho^\gamma=\fint_{B_R}\rho^{\gamma-1}(\rho-\rho^\infty)+\rho^\infty\fint_{B_R}\rho^{\gamma-1}\le\frac{C}{R^N}+\rho^\infty\left(\fint_{B_R}\rho^\gamma\right)^{\frac{\gamma-1}{\gamma}}\le\frac{C}{R^N}+\frac{\gamma-1}{\gamma}\fint_{B_R}\rho^\gamma+\frac{1}{\gamma}(\rho^\infty)^\gamma,$$

$$\fint_{B_R}\rho^\gamma\le(\rho^\infty)^\gamma+\frac{C\gamma}{R^N}.$$

   (b) ρ is bounded in $L^\infty+L^{2\gamma}$ when $N=3$.

$$\nabla\left(\rho^\gamma-\fint_{B_R}\rho^\gamma\right)=\rho f+g-\alpha\rho u-\mathrm{div}\ (\rho u\otimes u)+\mu\Delta u+\xi\nabla\mathrm{div}\ u=\nabla F+\mathrm{div}\ (\rho u\otimes u)+G$$

$$\left(F\in L^2,\ G=\alpha\rho u\in(L^\infty+L^\gamma)\cdot L^2\cap L^6=\left(L^{\frac{2\gamma}{\gamma+2}}+L^2\right)\cap\left(L^{\frac{6\gamma}{\gamma+6}}+L^6\right)=L^2\cap L^{\frac{6\gamma}{\gamma+6}}\text{ by (8)}\right),$$

$$\begin{aligned}\left\|\rho^\gamma-\fint_{B_R}\rho^\gamma\right\|_{2+6}&\le C\left(1+\|\rho u\otimes u\|_2\right)\\ &\le C\left(1+\|\rho\|_{\infty+2\gamma}+\left\||u|^2\right\|_{2\cap a}\right)\\ &\left(\frac{1}{2}=\frac{1}{2\gamma}+\frac{1}{a},\gamma>3\Rightarrow 1\le a<3\right),\end{aligned}$$

$$\|\rho\|_{\infty+2\gamma}^\gamma\le C\left(1+\|\rho\|_{\infty+2\gamma}\right)\ \left(L^{2\gamma}+L^{6\gamma}+L^\infty=L^{2\gamma}+L^\infty\right),$$

$$\|\rho\|_{\infty+2\gamma}\le C.$$

(c) $\rho-\rho^\infty$ is bounded in $L^2\cap L^{2\gamma}$, when $N=3$.

$$\int_{B_R}|\rho-\rho^\infty|^2=\int_{B_R}|\rho-\rho^\infty|^2\cdot 1_{\rho\leq 2\rho^\infty}+\int_{B_R}|\rho-\rho^\infty|^2\cdot 1_{\rho>2\rho^\infty}\leq\left(1+\int_{B_R}\rho^\gamma\cdot 1_{\rho>2\rho^\infty}\right)\leq C,$$

$$\begin{aligned}\int_{B_R}|\rho-\rho^\infty|^{2\gamma}&=\int_{B_R}|\rho-\rho^\infty|^{2\gamma}\cdot 1_{\rho\leq 2\rho^\infty}+\int_{B_R}|\rho-\rho^\infty|^{2\gamma}\cdot 1_{\rho>2\rho^\infty}\\&\leq C\left(\int_{B_R}|\rho-\rho^\infty|^2\cdot 1_{\rho\leq 2\rho^\infty}+\int_{B_R}\rho^{2\gamma}\cdot 1_{\rho>2\rho^\infty}\right)\leq C\ \left(\text{by (11)}\right).\end{aligned}$$

(d) ρ is bounded in $L^\infty+L^{\frac{N}{N-2}(\gamma-1)}$, $\rho-\rho^\infty$ is bounded in $L^2\cap L^{\frac{N}{N-2}(\gamma-1)}$, when $N=4$.

(e) $D\left\{\text{div } u-\frac{a}{\mu+\xi}\rho^\gamma\right\}$ is bounded in $\mathcal{H}^r_{loc}$, with $r\left\{\begin{array}{ll}<\infty, & \text{if } N=3,\\ =\frac{1}{q}+\left(\frac{1}{2}-\frac{1}{N}\right)+\frac{1}{2}, & \text{if } N\geq 4;\end{array}\right.$

$\text{div } u-\frac{a}{\mu+\xi}\rho^\gamma$ is compact in L^p_{loc}, with $p<\left\{\begin{array}{ll}\infty, & \text{if } N=3,\\ \frac{N}{N-2}\frac{\gamma-1}{\gamma}, & \text{if } N\geq 4.\end{array}\right.$

Step III: Passage to limit $R\to\infty$.

We show in this step

$$(\rho_{\alpha,R},u_{\alpha,R})\to_R(\rho_\alpha,u_\alpha),$$

by invoking [1, Theorem 6.4,Page 81]. The key point is to ensure

$$\int_{\mathbf{R}^N}\left\{\rho_\alpha-\overline{\rho_\alpha^\theta}^{\frac{1}{\theta}}\right\}u\cdot\nabla\varphi_n\to 0,\ \text{as } n\to\infty,$$

where φ is a cut-off function, $\varphi_n=\varphi\left(\frac{\cdot}{n}\right)$. This is in fact true since

$$\rho-\rho^\infty\in L^2!$$

Thus sending $R\to\infty$ in (12), we obtain a solution pair (ρ_α,u_α) of

$$\begin{cases}\alpha\rho+\text{div }(\rho u)=\alpha\rho^\infty, & \text{in } \mathbf{R}^N,\\ \alpha\rho u+\text{div }(\rho u\otimes u)-\mu\Delta u-\xi\nabla\text{div } u+a\nabla\rho^\gamma=\rho f+g, & \text{in } \mathbf{R}^N,\\ |u|\to 0, & \text{as } |x|\to\infty,\end{cases}\tag{13}$$

and the following energy inequality:

$$\int_{\mathbf{R}^N}\left\{\frac{\alpha}{2}\rho^\infty|u_\alpha|^2+\frac{\alpha}{2}\rho_\alpha|u_\alpha|^2+\mu|Du_\alpha|^2+\xi|\text{div } u_\alpha|^2+\frac{\alpha\gamma}{\gamma-1}\left(\rho_\alpha^\gamma-1-(\rho^\infty)^{\gamma-1}\right)(\rho_\alpha-\rho^\infty)\right\}$$

$$\leq \int_{\boldsymbol{R}^N} \{\rho_\alpha u_\alpha \cdot f + u_\alpha \cdot g\}.$$

Step IV: Uniform bounds independent of $\alpha \in (0, 1]$

This is done exactly the same as Step I. We omit the subscript α in the solution pair (ρ_α, u_α) in this step.

$$\|u\|_{\frac{2N}{N-2}} + \|Du\|_2 \leq C\left(1 + \|\rho\|_{\infty+q}\right);$$

$$\alpha \int_{\boldsymbol{R}^N} \left\{|u|^2 + \rho\,|u|^2\right\} \leq \left(\int_{\boldsymbol{R}^N} \rho\,|u|^2\right)^{1/2} \left(\int_{\boldsymbol{R}^N} \rho\,|f|^2\right)^{1/2} + \|u\|_2\,\|g\|_2$$
$$\Rightarrow \left(\alpha \int_{\boldsymbol{R}^N} \left\{|u|^2 + \rho\,|u|^2\right\}\right)^{1/2} \leq C\left(1 + \|\rho\|_{\infty+q}\right);$$

$$a\rho^\gamma = a\left(\rho^\infty\right)^\gamma + (\mu + \xi)\operatorname{div} u + R_i R_j\left(\rho u_i u_j\right) - (-\Delta)^{-1}(\rho f + g) - \alpha\,(-\Delta)^{-1} \operatorname{div}\ (\rho u)$$
$$\Rightarrow \|\rho\|_{\infty+q}^\gamma \leq C\left(1 + \|\rho\|_{\infty+q}^{\frac{2\gamma}{\gamma-1}}\right)$$
$$\left(\begin{array}{l} \alpha\,\|\rho\|_{2+s} \leq C\left(\alpha\left\|\sqrt{\rho}u\right\|_2\right)\|\rho\|_{\infty+q}^{1/2} \text{ with } \frac{1}{s} = \frac{1}{2} + \frac{1}{2q} \\ \frac{1}{s} - \frac{1}{N} < \frac{\gamma}{q} \Leftarrow \begin{cases} \frac{1}{2} - \frac{1}{N} = \frac{N-2}{2N} < \frac{2\gamma-1}{\gamma-1}\frac{N-2}{2N} = \frac{\gamma}{q} - \frac{1}{2q}, & \text{if } N \geq 4 \\ \frac{1}{2} + \frac{1}{2\cdot 2\gamma} - \frac{1}{3} < \frac{1}{2} = \frac{\gamma}{2\gamma}, & \text{if } N = 3 \end{cases} \end{array}\right)$$
$$\Rightarrow \begin{cases} \|\rho\|_{\infty+q} \leq C,\ \|u\|_{\frac{2N}{N-2}} \leq C,\ \|Du\|_2 \leq C, \\ \alpha\,\|u\|_2 \leq C, \alpha\,\|\rho u\|_{2+s} \leq C,\ \sqrt{\alpha}\,\|u\|_2 \leq C,\ \sqrt{\alpha}\left\|\sqrt{\rho}u\right\|_2 \leq C; \end{cases}$$
$$\Rightarrow \rho - \rho^\infty \text{ bounded in } L^1 + L^r \left(r = \begin{cases} 3, & \text{if } N = 3 \\ 2, & \text{if } N \geq 4 \end{cases}\right)$$
$$\Rightarrow \rho - \rho^\infty \text{ bounded in } \begin{cases} \left(L^1 + L^3\right) \cap L^\infty = L^3 \cap L^\infty, & \text{if } N = 3 \\ \left(L^1 + L^2\right) \cap \left(L^q + L^\infty\right) = L^2 \cap L^{q=\frac{N}{N-1}(\gamma-1)}, & \text{if } N \geq 4 \end{cases}$$

Step V: Passage to limit $\alpha \to 0_+$.

We now send $\alpha \to 0_+$ in the solution pair (ρ_α, u_α) of (13). Up to a subsequence, we may assume

$$\rho_\alpha \rightharpoonup \rho \text{ in } L^q_{loc},\ \text{with } \rho \in \begin{cases} L^3 \cap L^\infty, & \text{if } N = 3, \\ L^2 \cap L^q, & \text{if } N \geq 4; \end{cases}$$

$$u_\alpha \rightharpoonup u, \text{ in } L^{\frac{2N}{N-2}},\ u_\alpha \to u, \text{ in } L^p_{loc}\left(1 \le p < \frac{2N}{N-2}\right) \text{ with } u \in L^{\frac{2N}{N-2}},\ Du \in L^2;$$

$$\text{div } u_n - \frac{a}{\mu+\xi}\rho^\gamma \text{ converges strongly and a.e. in } L^p_{Loc}\left(1 \le p < \frac{q}{\gamma}\right).$$

Thus to invoke Theorem 6.4 in [1, Page 81] to see the strong convergence of ρ_α, we need to verify, by writing $r = \rho - \overline{\rho^\theta}^{\frac{1}{\theta}} \in \begin{cases} L^3 \cap L^\infty, & \text{if } N = 3, \\ L^2 \cap L^q, & \text{if } N \ge 4, \end{cases}$, that

$$\int_{\mathbf{R}^N} r\,|u \cdot \nabla\varphi_n| \to 0, \text{ as } n \to \infty, \tag{14}$$

where φ is a cut-off function, $\varphi_n = \varphi\left(\frac{\cdot}{n}\right)$.

1. Verification of (14) when $N \ge 4$.

$$\begin{aligned}\int_{\mathbf{R}^N} r\,|u \cdot \nabla\varphi_n| &\le \left(\int_{n\le|x|\le 2n} |r|^2\right)^{1/2} \cdot \left(\int_{n\le|x|\le 2n} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} \cdot \left(\int_{n\le|x|\le 2n}\right)^{1/N} \\ &\le C\left(\int_{n\le|x|\le 2n} |r|^2\right)^{1/2} \cdot \|u\|_{\frac{2N}{N-2}} \to 0, \text{ as } n \to \infty.\end{aligned}$$

2. Verification of (14) when $N = 3$.

   Noticing

$$\int_{\mathbf{R}^N} r\,|u \cdot \nabla\varphi_n| \le \left(\int_{n\le|x|\le 2n} |r|^3\right)^{1/3} \cdot \left(\int_{n\le|x|\le 6} |u|^6\right)^{\frac{1}{6}} \cdot \left(\int_{n\le|x|\le 2n}\right)^{1/2}, \tag{15}$$

   and $\frac{C}{n} \cdot n^{\frac{3}{2}} = Cn^{\frac{1}{2}}$, we need to find another way to overcome this difficulty. Recall from [1, Page 84],

$$0 \le \frac{1-\theta}{\theta}\frac{a}{\mu+\xi}\left\{\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma} \cdot \overline{\rho^\theta}\right\}\overline{\rho^\theta}^{\frac{1}{\theta}-1} \le \text{div }(ur).$$

   Thus multiplying the above inequality by φ_n, we get from (15) that

$$\int_{B_n} \left\{\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma} \cdot \overline{\rho^\theta}\right\}\overline{\rho^\theta}^{\frac{1}{\theta}-1} \le Cn^{\frac{1}{2}}.$$

   Taking $\theta = \frac{1}{2}$ and noticing that

$$\left.\begin{aligned} &f_n \rightharpoonup f,\ f_n^a \rightharpoonup \overline{f^b},\ f_n^b \rightharpoonup \overline{f^b} \\ &f_n = f_n^{1-\sigma} f_n^\sigma = (f_n^a)^{\frac{1-\sigma}{a}}\left(f_n^b\right)^{\frac{\sigma}{b}}\end{aligned}\right\} \Rightarrow \overline{f} \le \overline{f^a}^{\frac{1-\sigma}{a}} \cdot \overline{f^b}^{\frac{\sigma}{b}} \left(0 < \frac{1-\sigma}{a}, \frac{\sigma}{b} < 1\right),$$

we have

$$\left.\begin{array}{l} \rho = \overline{\rho} \le \overline{\rho^{\gamma+1/2}}^{\frac{1-\sigma}{\gamma+1/2}} \cdot \overline{\rho^{1/2}}^{\frac{\sigma}{1/2}} \\ \overline{\rho^{\gamma}} \le \overline{\rho^{\gamma+1/2}}^{\frac{\gamma(1-\tau)}{\gamma+1/2}} \cdot \overline{\rho^{1/2}}^{\frac{\gamma\tau}{1/2}} \\ \frac{1-\sigma}{\gamma+1/2} + \frac{\gamma(1-\tau)}{\gamma+1/2} = 1 = \frac{\sigma}{1/2} + \frac{\gamma\tau}{1/2} \end{array}\right\} \Rightarrow \rho\overline{\rho^{\gamma}} \le \overline{\rho^{\gamma+1/2}} \cdot \overline{\rho^{1/2}} \left(\sigma + \gamma\tau = \frac{1}{2}\right)$$

$$\Rightarrow \left\{\overline{\rho^{\gamma+1/2}} - \overline{\rho^{\gamma}} \cdot \overline{\rho^{1/2}}\right\}\overline{\rho^{1/2}} \ge \rho\overline{\rho^{\gamma}} - \overline{\rho^{\gamma}} \cdot \overline{\rho^{1/2}}^2 = \overline{\rho^{\gamma}}\left(\rho - \overline{\rho^{1/2}}^2\right) = \overline{\rho^{\gamma}} r$$

$$\Rightarrow \int_{B_{2n}} \overline{\rho^{\gamma}} r \le C n^{\frac{1}{2}}$$

$$\Rightarrow \left[\begin{array}{l} \int_{n\le|x|\le 2n} r \le C\left(\int_{n\le|x|\le 2n} \rho^{\gamma} r + \int_{n\le|x|\le 2n} \left|\overline{\rho^{\gamma}} - (\rho^{\infty})^{\gamma}\right| r^{\frac{1}{4}} r^{\frac{3}{4}}\right) \\ \qquad \le C\left(n^{\frac{1}{2}} + \left(\int_{n\le|x|\le 2n} \left|\overline{\rho^{\gamma}} - (\rho^{\infty})^{\gamma}\right|^2\right)^{1/2} \cdot \left(\int_{n\le|x|\le 2n} r\right)^{\frac{1}{4}} \cdot \|r\|_3^{\frac{3}{4}}\right) \end{array}\right.$$

$$\Rightarrow \int_{n\le|x|\le 2n} r \le C\left(1 + n^{\frac{1}{2}}\right) \le C n^{\frac{1}{2}}$$

$$\Rightarrow \int_{n\le|x|\le 2n} r\,|u\cdot\nabla\varphi_n| \le \frac{C}{n}\int_{n\le|x|\le 2n} r \le \frac{C}{n^{1/2}} \to 0, \text{ as } n\to\infty \ (u\in L^{\infty}).$$

2.2. (2) with small force f up to a gradient.

2.3. (2) with force f bounded by a normalization function.

3. **Stationary compressible Navier-Stokes equations in an exterior domain or a tube.**

4. **The "inflow" open problem.**

5. **Acknowledgements.** Thank all the brothers in the discussion group, for their patient suffering during my lectures. It is not easy to deliver all of [1, Sect. 6.8] in just three lectures. And hence some parts of this paper are left blank.

Happy Christmas!

REFERENCES

[1] P.L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models.* The Clarendon Press, Oxford University Press, New York, 1998.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com

Charleton University Mathematics Journal
Volume **1**, Number **15**, December **2010**

Website: http://www.sciencenet.cn/u/zjzhang/
pp. **69-75**

S.T. YAU COLLEGE MATHEMATICS CONTESTS 2010

ZUJIN ZHANG

ABSTRACT. In this paper, we give a reference answer to the Analysis and Differential Equations in S.T. Yau College Mathematics Contests 2010.

1. (a) Let $\{x_k\}_{k=1}^n \subset (0,\pi)$, and define

$$x = \frac{1}{n}\sum_{k=1}^n x_i.$$

Show that

$$\prod_{k=1}^n \frac{\sin x_k}{x_k} \le \left(\frac{\sin x}{x}\right)^n.$$

Proof. Direct computations show

$$\left(\ln\frac{\sin x}{x}\right)'' = (\ln\sin x - \ln x)'' = \frac{-1}{\sin^2 x} + \frac{1}{x^2} > 0,$$

for all $x \in (0,\pi)$. Thus $\ln\dfrac{\sin x}{x}$ is a convex function in $(0,\pi)$. Jensen's inequality then yields

$$\frac{1}{n}\sum_{k=1}^n \ln\frac{\sin x_k}{x_k} \le \ln\frac{\sin x}{x}.$$

The exponential of this above inequality is the desired result. □

(b) From

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\displaystyle\int_0^\infty \sin\left(x^2\right)\mathrm{d}x$.

Key words and phrases. S.T. Yau, College Mathematics Contests, analysis, differential equations.

Proof. Consider the sector in $\boldsymbol{R}^2$ enclosed by the following three curves

$$\begin{cases} I: & 0 \le z \le R, \\ II: & Re^{i\theta},\ 0 \le \theta \le \frac{\pi}{4}, \\ III: & re^{i\frac{\pi}{4}},\ 0 \le r \le R. \end{cases}$$

Cauchy's integration theorem then yields

$$0 = \left[\int_I + \int_{II} + \int_{III}\right] e^{iz^2}\mathrm{d}z. \tag{1}$$

Noticing

(i) $\displaystyle\int_I e^{iz^2}\mathrm{d}z = \int_0^R e^{ix^2}\mathrm{d}x$,

(ii)

$$\begin{aligned} \left|\int_{II} e^{iz^2}\mathrm{d}z\right| &= \left|\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}} \cdot iRe^{i\theta}\mathrm{d}\theta\right| \\ &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\mathrm{d}\theta \\ &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\cdot\frac{2}{\pi}\cdot 2\theta}\mathrm{d}\theta \\ &= \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \\ &\to 0, \text{ as } R \to \infty, \end{aligned}$$

(iii) $\displaystyle\int_{III} e^{iz^2}\mathrm{d}z = -\int_0^R e^{ir^2e^{i\frac{\pi}{2}}} \cdot e^{i\frac{\pi}{4}}\mathrm{d}r = e^{i\frac{\pi}{4}}\int_0^R e^{-r^2}\mathrm{d}r$,

we have, by sending $R \to \infty$ in (1), that

$$\int_0^\infty e^{ix^2}dx = e^{i\frac{\pi}{4}}\int_0^\infty e^{-r^2}\mathrm{d}r.$$

Taking the imaginary part of this above equality gives

$$\int_0^\infty \sin(x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

□

2. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be any function. Prove that the set

$$C = \left\{ x_0 \in \boldsymbol{R};\ f(x_0) = \lim_{x \to x_0} f(x) \right\}$$

is a G_δ-set.

Proof. By definition,

$$C = \cap_{k=1}^\infty C_k,$$

where

$$C_k = \left\{ x_0 \in \boldsymbol{R};\ \exists\, \delta_{x_0} > 0,\ s.t.\ |x - x_0| < \delta_{x_0} \Rightarrow |f(x) - f(x_0)| < \frac{1}{k} \right\}$$

is an open set. In fact,

$$x_0 \in C_k \Rightarrow U(x_0, \delta_{x_0}) \subset C_k.$$

□

3. Consider the *ODE*

$$\dot{x} = -x + f(t, x),$$

where

$$\begin{cases} |f(t,x)| \le \varphi(t)\,|x|\,,\ (t,x) \in \boldsymbol{R} \times \boldsymbol{R}, \\ \int^\infty \varphi(t)\mathrm{d}t < \infty. \end{cases}$$

Prove that every solution approaches zero as $t \to \infty$.

Proof. For all $t \in [0, \infty)$, we have

$$\infty \quad > \quad \int_0^t \varphi(s)\mathrm{d}s \ge \int_0^t \left| \frac{\dot{x}(s) + x(s)}{x(s)} \right| \mathrm{d}s = \int_0^t \left| \frac{(e^s x(s))'}{e^s x(s)} \right| \mathrm{d}s$$

$$\geq \left| \int_0^t \mathrm{d}\left(e^s x(s)\right) \right| = \left| e^t x(t) - x(0) \right|.$$

Thus

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-t} \cdot \left[e^t x(t)\right] = 0.$$

□

4. Solve the *PDE*

$$\begin{cases} \Delta u = 0, & \text{in } \boldsymbol{R}^+ \times \boldsymbol{R}, \\ u = g, & \text{on } \{x_1 = 0\} \times \boldsymbol{R}, \end{cases}$$

where

$$g(x_2) = \begin{cases} 1, & \text{if } x_2 > 0, \\ -1, & \text{if } x_2 < 0. \end{cases}$$

Proof. It is standard (easy to verfiy) that

$$u(x) = \int_{\{y_1=0\}\times\boldsymbol{R}} u(y) \frac{\partial G}{\partial \boldsymbol{n}}(x,y)\mathrm{d}S(y),$$

where

$$G(x,y) = \frac{1}{2\pi}\left[\ln|y-x| - \ln|y-\tilde{x}|\right]$$

is the Green's function for $\{x_1 > 0\}$, with $\tilde{x}$ the reflection of x in the plane $\{x_1 = 0\}$.

Direct computations show

$$\begin{aligned} \frac{\partial G}{\partial \boldsymbol{n}}(x,y) &= -\frac{\partial G}{\partial y_1}(x,y) = -\frac{1}{2\pi}\left[\frac{y_1 - x_1}{|y-x|^2} - \frac{y_1 + x_1}{|y-\tilde{x}|}\right] \\ &= -\frac{1}{2\pi}\frac{-2x_1}{|y-x_1|^2} \quad (|y-x| = |y-\tilde{x}|) \\ &= \frac{x_1}{\pi|y-x|^2}. \end{aligned}$$

Thus

$$\begin{aligned}
u(x) &= \int_{\{y_1=0\}\times \boldsymbol{R}} u(y)\frac{x_1}{\pi\,|y-x|^2}\mathrm{d}S(y) \\
&= -\frac{x_1}{\pi}\int_{-\infty}^{\infty}\frac{g(y_2)}{x_1^2+(y_2-x_2)^2}\mathrm{d}y_2 \\
&= -\frac{x_1}{\pi}\left[\frac{1}{x_1}\int_{-\infty}^{0}\frac{-1}{1+\left|\frac{y_2-x_2}{x_1}\right|^2}\mathrm{d}\frac{y_2-x_2}{x_1}+\frac{1}{x_1}\int_{0}^{\infty}\frac{1}{1+\left(\frac{y_2-x_2}{x_1}\right)^2}\mathrm{d}\frac{y_2-x_2}{x_1}\right] \\
&= -\frac{1}{\pi}\left[-\arctan\frac{y_2-x_2}{x_1}\bigg|_{y_2=-\infty}^{y_2=0}+\arctan\frac{y_2-x_2}{x_1}\bigg|_{y_2=0}^{y_2=\infty}\right] \\
&= \frac{2}{\pi}\arctan\frac{x_2}{x_1},\ x=(x_1,x_2)\in \boldsymbol{R}^2.
\end{aligned}$$

□

5. Let $K\in C\left([0,1]\times[0,1]\right)$. For $f\in C[0,1]$, the space of continuous functions on $[0,1]$, define

$$Tf(x)=\int_0^1 K(x,y)f(y)\mathrm{d}y.$$

Prove that $Tf\in C[0,1]$. Moreover,

$$\Omega=\left\{Tf;\ \|f\|_{sup}\le 1\right\}$$

is precompact in $C[0,1]$.

Proof. (a) $Tf\in C[0,1]$.

$$\begin{aligned}
|Tf(x_1)-Tf(x_2)| &\le \int_0^1 |K(x_1,y)-K(x_2,y)|\,|f(y)|\,\mathrm{d}y \\
&\to 0,\ \text{as}\ |x_1-x_2|\to 0, \qquad (2)
\end{aligned}$$

by the uniform continuity of K in x and y.

(b) Ω is precompact in $C[0,1]$.

This follows readily from

(i) the unform boundedness of $f \in \Omega$:

$$\|f\|_{\sup} \leq 1,$$

(ii) the equicontinuity of $f \in \Omega$, that is, (2),

(iii) and the Ascoli-Azerá theorem.

$\square$

6. Prove the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(x+2n\pi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

for

$$f \in \mathcal{S}(\boldsymbol{R}) = \left\{ f \in L^1_{loc}(\boldsymbol{R});\ (1+|x|^m) \left| f^{(n)}(x) \right| \leq C_{m,n},\ \forall\, m,n \geq 0 \right\}.$$

Here

$$\hat{f}(\xi) = \int_{\boldsymbol{R}} f(x) e^{-ix\xi} \mathrm{d}x.$$

Proof. Define

$$h(x) = \sum_{n=-\infty}^{\infty} f(x+2n\pi).$$

Then h is periodic with periodical 2π. And hence the coefficients of its Fourier series are

$$\begin{aligned}
a_k &= \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-ikx} \mathrm{d}x = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} f(x+2n\pi) e^{-ikx} \mathrm{d}x \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{2(n+1)\pi} f(y) e^{-ik(y-2n\pi)} \mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(x) e^{-ikx} \mathrm{d}x = \hat{f}(k).
\end{aligned}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} f(x+2n\pi) = h(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}.$$

□

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA

E-mail address: uia.china@gmail.com